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Classical and Quantum Algebraic Screening in a Coulomb Plasma near a Wall: A Solvable Model

J. N. Aqua[1] **and F. Cornu**[2]

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The static position correlation in a quantum Coulomb plasma near a wall is studied by means of a model where two quantum charges are embedded in a classical plasma at equilibrium. Three kinds of walls are considered: a wall without electrostatic properties, a dielectric, and an ideal conductor. At large separations y along the wall, the correlation exactly decays as $1/y^3$, though no algebraic tail exists for classical charges near an ideal conductor. This tail originates from thermal statistical and purely quantum fluctuations of polarization clouds which are deformed by the geometric constraint due to the wall and by the charges induced by influence inside a wall with electrical properties. The coefficient of the $1/y^3$ tail can be calculated explicitly in a weak-coupling and low-delocalization regime. Then classical, diffraction, and purely quantum contributions are disentangled.

KEY WORDS: Coulomb systems; quantum screening; surface correlations; solvable model.

1. INTRODUCTION

1.1. The Issue at Stake

At the scale of condensed matter physics, the sole relevant interaction between fundamental particles (such as electrons, nuclei, atoms, or ions) is the electrostatic interaction. The Coulomb potential $v_C(\mathbf{r}, \mathbf{r}')$ created at $\mathbf{r}'$ by a unit point charge located at $\mathbf{r}$ is the solution of the Poisson equation,

$$\Delta_{\mathbf{r}'} v_C(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}') \tag{1.1}$$

[1] Laboratoire de Physique (Laboratoire Associé au C.N.R.S. URA 1325), Ecole Normale Supérieure de Lyon, F-69364 Lyon Cedex 07, France.

[2] Laboratoire de Physique Théorique (Unité Mixte de Recherche du C.N.R.S. UMR 8627), Université de Paris-Sud, F-91405 Orsay Cedex, France.

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Equation (1.1) is written in Gauss units in three dimensions. The expression of the long-ranged potential $v_C(\mathbf{r}, \mathbf{r}')$ depends on the boundary conditions for the domain in which the charges are allowed to move. For instance, in the bulk of the system, the microscopic pair potential is merely the value of the Coulomb potential between two unit charges in the vacuum,

$$v_{C\text{bulk}}(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{1.2}$$

However, $v_C(\mathbf{r}, \mathbf{r}')$ takes a different form in the vicinity of a wall with electrical properties. These are characterized by the dielectric constant ε_W of the wall material. However, some strong screening, one of the most fundamental properties in Coulomb plasmas at equilibrium, arises for any particular solution of Eq. (1.1), namely for any boundary conditions, as exemplified below.

More precisely, apart from screening effects originating from the long range of the interaction between elementary charges—such as the absence of any volumic charge required for the stability of a macroscopic system—some features of microscopic screening, such as the large-distance behavior of correlations, are specific to the fact that $v_C(\mathbf{r}, \mathbf{r}')$ is the solution of Eq. (1.1). In the bulk, the specificity is very clear. When particles obey classical dynamics, correlations fall off exponentially fast at large distances $|\mathbf{r} - \mathbf{r}'|$ in the Coulomb,[1] whereas they behave only algebraically for any other generic nonintegrable force.[2] When quantum dynamics is taken into account (with either Maxwell–Boltzmann[3] or Fermi–Bose[4] statistics), Coulomb screening becomes algebraic; nevertheless, the $1/|\mathbf{r} - \mathbf{r}'|^6$ power law of the correlation decay is faster than it would be if the potential were not harmonic (see Section III B of ref. 4).

In the vicinity of a wall, the specificity with respect to other long-ranged potentials is partially canceled by the breakdown of rotational invariance. In the *classical* regime, the static position correlation at large distances when particles remain near the wall at fixed distances x and x' decays as

$$\frac{f_{\text{cl}}(x, x'; \varepsilon_W)}{y^3} \qquad \text{when } \varepsilon_W \text{ is finite} \tag{1.3}$$

In Eq. (1.3) y is the norm of the projection of $\mathbf{r} - \mathbf{r}'$ in the plane parallel to the wall. The interpretation is that there is still internal screening—namely, the global charge of a particle and its polarization cloud vanishes after statistical averaging—but in the vicinity of the wall the mean dipole carried by a charge and its polarization cloud does not vanish in the direction perpendicular to the boundary; subsequently effective dipole–dipole interactions appear. On the contrary, when the wall is an ideal conductor ($\varepsilon_W = \infty$), the correlation along the wall falls off faster than any inverse power law. The reason is that some "perfect" influence phenomenon inside the wall balances the loss of rotational symmetry inside the plasma.[5] In the *quantum* regime, when ε_W is finite, the surface–charge correlation, namely the charge correlation after integration over the distances x and x' from the wall, also decays as $1/y^3$ in the special case of a One-Component Plasma (OCP) where one species of moving charges is embedded in a rigid neutralizing background. According to ref. 6, this is a consequence of the fluctuation-dissipation theorem and the assumption that the charge induced in the plasma by an infinitesimal time-dependent external charge uniformally spread on the wall surface is localized over a microscopic scale from the wall. As for the particle-particle correlation (without integration over x or x'), it is also expected to decrease as $1/y^3$. Indeed, on one hand, internal screening also works in a quantum plasma so that effective interactions decrease as or faster than $1/y^3$. On the other hand, quantum screening is less efficient than the classical one—because of intrinsic quantum fluctuations of position for any particle—so that the quantum correlation should decay as or more slowly than $1/y^3$.

In short, at equilibrium, the various behaviors of static position correlations at large distances when particles remain in the vicinity of the wall mainly depend on three properties:

(I) the electrostatic response of the wall, which determines the form of the microscopic potential $v_C(\mathbf{r}, \mathbf{r}')$ through its dependence upon the dielectric constant ε_W of the wall,

(II) the classical or quantum nature of thermal fluctuations,

(III) the geometrical constraints imposed to those fluctuations near the wall.

The aim of the present paper is to investigate how these effects interplay in the quantum regime by means of a simple solvable model: two external quantum charges embedded in a classical plasma at thermal equilibrium in the vicinity of a wall. (The classical bath may contain a rigid electrical background). We consider three kinds of walls: a wall without any electrostatic property ($\varepsilon_W = 1$), a dielectric material ($0 < \varepsilon_W < 1$ or $1 < \varepsilon_W < \infty$) and an ideal conductor ($\varepsilon_W = \infty$). These walls carry no external charge (but contain internal polarization charge when $\varepsilon_W \neq 1$). The model in the case of two quantum charges in the bulk was first introduced by Alastuey and Martin[3] in order to exhibit the origin of intrinsically algebraic quantum screening in the plasma bulk. This origin is conveniently dealt with by using the Feynman–Kac path integral representation of the quantum Gibbs factor. Indeed, the random paths of the formulation are a proper way for describing the consequences of quantum fluctuations upon large-distance quantities.

1.2. Results

The first interest of our statistical mechanical model is that it is exactly solvable. The result of the calculation is that, at the inverse temperature $\beta \equiv 1/k_B T$, where k_B is Boltzmann constant, the effective interaction $\Phi^{\mathrm{ext}}_{\alpha_1\alpha_2}(x_1, x_2, y)$ between two *external* quantum charges (of species α_1 and α_2) embedded in a classical plasma decays as $1/y^3$ along the wall,

$$-\beta\Phi^{\mathrm{ext}}_{\alpha_1\alpha_2}(x_1, x_2, y) \sim_{y\to\infty} \frac{f(x_1, x_2; \varepsilon_W)}{y^3} \tag{1.4}$$

Equation (1.4) is valid for any value of ε_W ($\varepsilon_W = \infty$ included) and for all values of the parameters temperature T and densities ρ_j of the various species j in the classical plasma in a fluid phase.

A second interest of the model is that it is expected to capture the main features of a fully quantum system in a regime where exchange effects are negligible, while some intricate details of the quantum many-body problem, which are not crucial for screening phenomenon, are avoided.[7] From this point of view, the result of the model strengthens the conjecture of Section 1.1 based on general arguments: in a quantum plasma the static Ursell function, defined as minus 1 plus the probability of finding a particle at $\mathbf{r}'$ when there is a particle at $\mathbf{r}$, also has a

$$\frac{f_{\mathrm{qu}}(x, x'; \varepsilon_W)}{y^3} \tag{1.5}$$

tail for any value of ε_W and even in the whole fluid phase (i.e. in a wide range of temperature and densities). This tail is the superposition of a quasi–electrostatic and geometric effect $f_{\mathrm{qu,\,elect}}(x, x'; \varepsilon_W)/y^3$, which also arises in the classical regime except when the wall is an ideal conductor, and of a purely quantum contribution $f_{\mathrm{qu},\,\omega}(x, x'; \varepsilon_W)/y^3$, which exists for any value of ε_W,

$$f_{\mathrm{qu}}(x, x'; \varepsilon_W) = f_{\mathrm{qu,\,elect}}(x, x'; \varepsilon_W) + f_{\mathrm{qu},\,\omega}(x, x'; \varepsilon_W) \tag{1.6}$$

with

$$f_{\mathrm{qu,\,elect}}(x, x'; \varepsilon_W = \infty) = 0 \tag{1.7}$$

Third, the model provides a picture of the profile amplitude $f_{\mathrm{qu}}(x, x'; \varepsilon_W)$ for the expected $1/y^3$ tail of the Ursell function in the fully quantum problem. Indeed, the general expression of the $f(x_1, x_2, \varepsilon_W)/y^3$ tail in the model can be made explicit in a regime where the Coulomb coupling is weak and the quantum charges are slightly delocalized. In this double limit, the electrostatic contribution $f_{\mathrm{elect}}(x_1, x_2; \varepsilon_W)$ in the model appears as the sum of a classical contribution plus a "diffraction" correction due to quantum dynamics. Hereafter, the term diffraction will refer to effects which are only corrections to the classical behavior and arise from the wave nature of quantum dynamics. We infer that similarly in the quantum many-body problem $f_{\mathrm{qu, elect}}(x, x'; \varepsilon_W)$ decreases at least faster than any inverse power-law over a distance ξ from the wall, where the Debye length ξ of the quantum plasma is the length scale for classical Coulomb screening,

$$\xi \equiv \frac{1}{\sqrt{4\pi\beta \sum_\gamma \rho_\gamma e_\gamma^2}} \tag{1.8}$$

(In Eq. (1.8) e_γ (ρ_γ) is the charge (density) of moving species γ). On the contrary, from the study of the profile $f_{\mathrm{qu}, \omega}(x_1, x_2; \varepsilon_W)$ in the model, we expect that the purely quantum contribution $f_{\mathrm{qu}, \omega}(x, x'; \varepsilon_W)$ is concentrated over a density-independent scale λ from the wall, λ being the order of magnitude of the thermal de Broglie wavelengths λ_γ,

$$\lambda_\gamma \equiv \sqrt{\beta \hbar^2/m_\gamma} \tag{1.9}$$

In Eq. (1.9) $\hbar$ is Planck constant and m_γ is the mass of species γ. The "electrostatic" part of the effective A/y^3 interaction between species γ and γ' per unit area is equal to $(-1/\beta)$ times $\int_0^\infty dx \int_0^\infty dx' \rho_\gamma(x)\, \rho_{\gamma'}(x')\, f_{\mathrm{qu, elect}}(x, x'; \varepsilon_W)/y^3$. The order of magnitude of its coefficient is equal to its classical value $1/(\beta^2 e_\gamma e_{\gamma'})$ when ε_W is finite. On the other hand, $f_{\mathrm{qu}, \omega}(x, x'; \varepsilon_W)$ starts at order $\lambda^2 \propto \hbar^2$, because λ is the amplitude of the quantum position fluctuations confined by the wall and which generates $f_{\mathrm{qu}, \omega}(x, x'; \varepsilon_W)$. The amplitude of the corresponding part in the $1/y^3$ effective interaction between surface charges (after integration over x and x') is expected to be of order $[1/(\beta^2 e_\gamma e_{\gamma'})] \times (\lambda/\xi)^4$, because $f_{\mathrm{qu}, \omega}$ is localized inside the plasma over a width of order λ.

We notice that $f_{\mathrm{qu}, \omega}(x, x'; \varepsilon_W)/y^3$ may be interpreted as $-\beta$ times a repulsive potential.

The paper is organized as follows. The formal resolution presented in Section 2 is valid for various situations (in the bulk, or near a boundary, or—with a slight alteration—in the presence of a uniform magnetic field[8]). The details are recalled in order to introduce the notations. In Section 3 an explicit formula for the electrostatic interaction between the random paths introduced by the Feynman–Kac representation is given in the weak-coupling regime. The latter is reached when

$$a \ll \xi_B \tag{1.10}$$

where a is the mean distance between particles in the classical plasma and between those and the external charges, while ξ_B is the Debye length of the classical bath

$$\xi_B \equiv \frac{1}{\sqrt{4\pi\beta \sum_j \rho_j Q_j^2}} \tag{1.11}$$

(In Eq. (1.11) the sum runs over the species of moving particles in the classical bath, each of which has a density ρ_j and a charge Q_j.) Moreover (Section 4.1), the path integrals may be performed explicitly in the low-delocalization limit. The latter is define as the regime where the averaged extent λ_i of position fluctuations for each quantum particle is negligible compared with the screening length of the classical plasma,

$$\lambda_i \ll \xi_B \tag{1.12}$$

We get the exact value of $f_\omega(x_1, x_2; \varepsilon_W = \infty)$ in the double limit (1.10) and (1.12). The leading term is of order $\hbar^2$. When ε_W is finite, the electrostatic contribution $f_{\text{elect}}(x_1, x_2; \varepsilon_W)$ can be calculated up to order $\hbar^2$ in the more restricted regime $a \ll \lambda_i \ll \xi_B$ (Section 4.2). Thus a quantitative comparison between the $1/y^3$ tails from various origins is allowed (Section 4.3). Section 5 is devoted to an attempt for the description of the correlation in a quantum OCP in the vicinity of an ideally conducting wall. In this case a phenomenological model for the fully quantum system in the low-density regime is introduced from the result of the solvable model. The low-density regime corresponds to a low-degeneracy and weak-coupling limit,

$$\lambda \ll a \ll \xi \tag{1.13}$$

Then $1/y^3$ tail for the phenomenological quantum correlation when $\varepsilon_W = \infty$ becomes larger than the exponentially-decaying classical term only at distances of order ten ξ. As a conclusion (Section 6), the behavior of static position correlations may be interpreted in terms of effective dipole–dipole interactions between the global entities made by any charge and its fluctuating polarization cloud.

2. THE GENERAL MODEL

2.1. Definitions

We consider two quantum point charges of species α_i characterized by their charges e_i and their masses m_i and which are embedded in a classical plasma. The latter may be made of several species of point charges and a possible uniform rigid electrostatic background. The masses of classical particles do not appear in the static equilibrium averages. Their charges are of the same order of magnitude as those of the two quantum particles. A wall with dielectric constant ε_W confines all particles to the semi-infinite three-dimensional domain $x > 0$.

The Coulomb potential v_{CW} between two charges in the vicinity of the wall is the solution of the Poisson Eq. (1.1) with the following boundary conditions. In all cases v_{CW} vanishes when x goes to $\pm\infty$ while v_{CW} and the normal component of the electric displacement $\varepsilon\,\nabla v_{CW}$ are continuous across the boundary plane $x = 0$. (When $\varepsilon_W = \infty$, the latter condition means that v_{CW} vanishes inside the ideally conductive wall.) The solution may be written in one and the same form, for any value of ε_W (even for $\varepsilon_W = \infty$),

$$v_{CW}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} + \frac{1-\varepsilon_W}{1+\varepsilon_W}\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2^*|} \qquad \text{for} \quad x_1 > 0 \quad \text{and} \quad x_2 > 0 \tag{2.1}$$

In Eq. (2.1) $\mathbf{r}^* = (-x, y, z)$ is the image of $\mathbf{r} = (x, y, z)$ with respect to the plane $x = 0$. In the case $\varepsilon_W = 1$, $v_{CW}(\mathbf{r}_1, \mathbf{R}_2)$ is reduced to its bulk (or vacuum) value (1.2): the wall has no electrostatic property. Such a wall is often called a "plain" hard wall in the literature. When $\varepsilon_W \neq 1$, v_{CW} may be interpreted as the sum of its bulk value $v_{C\text{bulk}}$ plus the contribution from an "image". The image of a unit charge at $\mathbf{r}$ is a particle carrying a charge $(1-\varepsilon_W)/(1+\varepsilon_W)$ and located at $\mathbf{r}^*$. The image charge has the same effect as the charges which appear by influence inside the wall in the presence of a charge outside the wall.

In the classical system, a short-ranged repulsive potential v_{SR} must be introduced in order to prevent the two-body collapse between classical charges with opposite signs. [In the quantum case the collapse is avoided by the uncertainty principle.] The collapsing particles may be either true particles or, when $\varepsilon_W > 1$ ($\varepsilon_W < 1$), they may be a charge at $\mathbf{R}_1$ and the image at $\mathbf{R}_2^*$ of another charge with the same (opposite) sign. Both situations arise in a multicomponent plasma made of moving charges of both signs, while only the second kind of "dangerous" attraction appears in the case of a One-Component Plasma. We recall that the latter system is made of one species of moving charges e and a continuous rigid neutralizing background whose uniform charge density is exactly opposite to the mean bulk charge density of moving charges. (v_{SR} can be suppressed only in the case of a One-Component plasma near a plain wall or when $\varepsilon_W < 1$.) The short-distance repulsion v_{SR} ensures that the thermodynamical quantities are well-defined, while it has no effect on large-distance screening properties which are entirely ruled by v_{CW}.

The sum of all interactions when the two quantum particles are at $\mathbf{r}_1$ and $\mathbf{r}_2$ and the classical charges are in the position configuration $\mathscr{C}$ is $U_W(x_1) + U_W(x_2) + U_{\text{tot}}(\mathbf{r}_1, \mathbf{r}_2, \mathscr{C})$ where $U_W(x)$ is a repulsive potential describing an infinitely steep wall,

$$\exp[-\beta U_W(x)] = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x < 0 \end{cases} \tag{2.2}$$

$U_{\text{tot}}(\mathbf{r}_1, \mathbf{r}_2, \mathscr{C})$ may be decomposed into

$$U_{\text{tot}}(\mathbf{r}_1, \mathbf{r}_2, \mathscr{C}) = e_1 e_2 v_{CW}(\mathbf{r}_1, \mathbf{r}_2) + \sum_{i=1,2} V_i(\mathbf{r}_i, \mathscr{C}) + U_0(\mathscr{C}) \tag{2.3}$$

where V_i is the interaction between each quantum particle with index i and the plasma,

$$V_i(\mathbf{r}_i, \mathscr{C}) \equiv e_i \int d\mathbf{r}\, v_{CW}(\mathbf{r}_i, \mathbf{r})\, Q(\mathbf{r}) \tag{2.4}$$

while U_0 is the energy of the classical plasma in the absence of any quantum charge

$$U_0(\mathscr{C}) \equiv \int d\mathbf{r}\, U_W(x)\, \rho(\mathbf{r}) + \frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}'\, Q(\mathbf{r})[v_{CW}(\mathbf{r}, \mathbf{r}') + v_{SR}(\mathbf{r}, \mathbf{r}')]\, Q(\mathbf{r}') \tag{2.5}$$

In the above definitions $\rho(\mathbf{r})$ ($Q(\mathbf{r})$) is the particle (charge) density operator, $\rho(\mathbf{r}) \equiv \sum_j \delta(\mathbf{r} - \mathbf{r}_j)$ and

$$Q(\mathbf{r}) \equiv \sum_j Q_j \delta(\mathbf{r} - \mathbf{R}_j) + Q_B(x) \tag{2.6}$$

where $\mathbf{R}_j$ (Q_j) is the position (charge) of the classical particle indexed by j and $Q_B(x)$ is the charge density of the possible uniform background in the half-space $x > 0$.

The effective interaction $\Phi^{\text{ext}}_{\alpha_1\alpha_2}(\mathbf{r}_1, \mathbf{r}_2)$ between the two quantum particles is defined from the immersion free energies $F^{(1)}_{\alpha_i}(\mathbf{r}_i)$ (or $F^{(2)}_{\alpha_1\alpha_2}(\mathbf{r}_1, \mathbf{r}_2)$) of one (or two) particle(s) in the classical plasma through

$$\Phi^{\text{ext}}_{\alpha_1\alpha_2}(\mathbf{r}_1, \mathbf{r}_2) \equiv F^{(2)}_{\alpha_1\alpha_2}(\mathbf{r}_1, \mathbf{r}_2) - F^{(1)}_{\alpha_1}(\mathbf{r}_1) - F^{(1)}_{\alpha_2}(\mathbf{r}_2) \tag{2.7}$$

The free energies are calculated from statistical averages of the corresponding Gibbs factors, where the averages are performed over the states of the classical plasma. These thermal averages are denoted by $\langle \cdots \rangle_{U_0}$. Indeed, when dynamics is classical, the contributions to the Gibbs factor from the kinetic and potential parts of the Hamiltonian factorize; then the thermal average $\langle A \rangle_{U_0}$ of an observable A which does not depend on the momenta of classical particles involves only U_0 in any statistical (canonical or grand canonical) ensemble. For instance in the canonical ensemble

$$\langle A \rangle_{U_0} \equiv \frac{\int d\mathscr{C}\, A \exp[-\beta U_0(\mathscr{C})]}{\int d\mathscr{C} \exp[-\beta U_0(\mathscr{C})]} \tag{2.8}$$

where $\int d\mathscr{C}$ denotes an integration over the position configurations $\mathscr{C}$ of plasma particles. If $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_2$ denote the momentum *operators* of the two quantum particles, the free energies are defined respectively through

$$e^{-\beta F^{(1)}_{\alpha_i}(\mathbf{r}_i)} = \left\langle \langle \mathbf{r}_i | \exp\left\{ -\beta \left[\frac{\hat{\mathbf{p}}_i^2}{2m_i} + U_W(\hat{x}_i) + V_i(\hat{\mathbf{r}}_i, \mathscr{C}) \right] \right\} |\mathbf{r}_i\rangle \right\rangle_{U_0} \tag{2.9}$$

and

$$e^{-\beta F^{(2)}_{\alpha_1\alpha_2}(\mathbf{r}_1, \mathbf{r}_2)} = \left\langle \langle \mathbf{r}_1 \mathbf{r}_2 | \exp\left\{ -\beta \left[\frac{\hat{\mathbf{p}}_1^2}{2m_1} + \frac{\hat{\mathbf{p}}_2^2}{2m_2} + U_W(\hat{x}_1) + U_W(\hat{x}_2) + e_1 e_2 v_{CW}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) + \sum_{i=1,2} V_i(\hat{\mathbf{r}}_i, \mathscr{C}) \right] \right\} |\mathbf{r}_1 \mathbf{r}_2\rangle \right\rangle_{U_0} \tag{2.10}$$

(We do not take into account the exchange matrix-element for the Gibbs factor, because it is expected to decay faster than the direct matrix-element when $|\mathbf{r}_1 - \mathbf{r}_2|$ goes to infinity and we are interested only in large $|\mathbf{r}_1 - \mathbf{r}_2|$ effects.) We already notice that symmetry arguments imply that $\Phi^{\text{ext}}_{\alpha_1\alpha_2}(\mathbf{r}_1, \mathbf{r}_2) = \Phi^{\text{ext}}_{\alpha_1\alpha_2}(x_1, x_2, \mathbf{y})$, with the notations of the Introduction.

2.2. Formal Resolution

2.2.1. Feynman–Kac Formula. The matrix elements of the quantum Gibbs factor may be expressed as some kinds of classical Gibbs factors by using the Feynman–Kac formula. The difficulty originating from the non commutativity of the position and momentum operators is then replaced by the task of integrating over Brownian paths. Let us introduce the Brownian dimensionless bridges $\xi_i(s)$ where s is a dimensionless abscissa with $0 \leqslant s \leqslant 1$; $\xi_i(s=0) = \xi_i(s=1) = \mathbf{0}$. Then the Feynman–Kac formula reads[9]

$$\langle \mathbf{r}_1 \mathbf{r}_2 | \exp\left\{ -\beta \left[\frac{\hat{\mathbf{p}}_1^2}{2m_1} + \frac{\hat{\mathbf{p}}_2^2}{2m_2} + U_W(\hat{x}_1) + U_W(\hat{x}_2) + U_{\text{tot}}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \mathscr{C}) \right] \right\} | \mathbf{r}_1 \mathbf{r}_2 \rangle$$
$$= \frac{1}{(2\pi\lambda_1^2)^{3/2}} \frac{1}{(2\pi\lambda_2^2)^{3/2}} \int D_W^0(\xi_1; x_1) \int D_W^0(\xi_2; x_2)$$
$$\times \exp\left[-\beta \int_0^1 ds\, U_{\text{tot}}(\mathbf{r}_1 + \lambda_1 \xi_1(s), \mathbf{r}_2 + \lambda_2 \xi_2(s), \mathscr{C}) \right] \tag{2.11}$$

In (2.11) the de Broglie wavelength $\lambda_i \equiv \hbar \sqrt{\beta / m_i}$ measures the amplitude of quantum position fluctuations at inverse temperature β. The potential $U_W(x)$, which describes the impenetrability of the wall, is entirely taken into account in the measure $D_W^0(\xi; x)$ for a dimensionless Brownian bridge confined to a semi-infinite space and with its origin $\mathbf{r}$ at a distance x from the wall.

Explicit results may be obtained with the measure $D_W^0(\xi; x)$. For instance, according to the one-body analog of (2.11), this measure is normalized by

$$\frac{1}{(2\pi\lambda_i^2)^{3/2}} \int D_W^0(\xi_i; x_i) \equiv \langle \mathbf{r}_i | \exp\left\{ -\beta \left[\frac{\hat{\mathbf{p}}_i^2}{2m_i} + U_W(\hat{x}_i) \right] \right\} | \mathbf{r}_i \rangle \tag{2.12}$$

The free thermal propagator $\langle \mathbf{r}' | \exp\{ -\beta [\hat{\mathbf{p}}_i^2 / (2m_i) + U_W(x_i)]\} | \mathbf{r} \rangle$ is the product of three one-dimensional free propagators. In the direction x perpendicular to the wall, the propagator is

$$g_{0,W}(x, x'; s) \equiv \langle x | \exp\{ -\beta s\, [\hat{h}^0 + U_W(\hat{x})]\} | x' \rangle$$
$$= \theta(x)\, \theta(x') \frac{1}{(2\pi\lambda^2)^{1/2}}$$
$$\times \{ \exp[-|x - x'|^2 / 2\lambda^2 s] - \exp[-|x + x'|^2 / 2\lambda^2 s] \} \tag{2.13}$$

where $\langle x | \beta \hat{h}^0 \equiv -(\lambda^2/2)\, \partial^2 / (\partial x)^2$ and $\theta(x)$ is the Heaviside function defined as $\theta(x) = 0$ if $x < 0$, $\theta(x=0) = 1/2$ and $\theta(x) = 1$ if $x > 0$. $g_{0,W}(x, x'; s)$ vanishes when x or x' tends to zero, in agreement with the fact that the probability amplitude of finding a particle in an impenetrable wall vanishes. With the latter notation, (2.12) is reduced to

$$\frac{1}{(2\pi\lambda_i^2)^{3/2}} \int D_W^0(\xi_i; x_i) = \frac{1}{(2\pi\lambda_i^2)} g_{0,W}(x_i, x_i; s=1)$$
$$= \frac{1}{(2\pi\lambda_i^2)^{3/2}} \theta(x_i)[1 - e^{-2x_i^2/\lambda_i^2}] \tag{2.14}$$

We notice that the set of Brownian bridges originating at x has a vanishing measure as x comes close to the wall. The first and second moments $\int D_W^0(\xi_i; x_i)[\xi(s)]_x$ and $\int D_W^0(\xi_i; x_i)[\xi(s)]_x^2$ also tend to zero as x_i^2 in the same limit, as it can be checked in the formulae given below.

2.2.2. Difference with Electrostatic Free Energies. In the case of the one-body matrix element of the Gibbs factor involved in (2.9), a Feynman–Kac formula similar to (2.11) allows one to get an expression only in terms of some electrostatic interaction. Indeed, let us define

$$q_i(\mathbf{r}) \equiv e_i \int_0^1 ds\, \delta(\mathbf{r} - \mathbf{r}_i - \lambda_i \xi_i(s)) \tag{2.15}$$

$q_i(\mathbf{r})$ is a charge distribution uniformally spread over a wire with the same shape as the Brownian bridge $\lambda_i \xi_i$ with its origin at $\mathbf{r}_i$. The contribution from $V_i(\mathbf{r}_i, \mathscr{C})$ to the path integral may be written as the electrostatic energy between the charge distribution $q_i(\mathbf{r})$ and the configuration $\mathscr{C}$ of the classical plasma,

$$\int_0^1 ds\, V_i(\mathbf{r}_i + \lambda_i \xi_i(s), \mathscr{C}) = \int d\mathbf{r} \int d\mathbf{r}'\, q_i(\mathbf{r})\, v_{CW}(\mathbf{r}, \mathbf{r}')\, Q(\mathbf{r}') \equiv V[q_i] \tag{2.16}$$

Henceforth the immersion free energy $F_{\alpha_i}^{(1)}(\mathbf{r}_i)$ of one quantum particle is related to the corresponding free energy $F_{\text{elect}}^{(1)}[q_i]$ for a classical extended wire which interacts with the classical plasma through the electrostatic interaction. The relation is

$$e^{-\beta F_{\alpha_i}^{(1)}(\mathbf{r}_i)} = \frac{1}{(2\pi\lambda_i^2)^{3/2}} \int D_W^0(\xi_i; x_i)\, e^{-\beta F_{\text{elect}}^{(1)}[q_i]} \tag{2.17}$$

with

$$e^{-\beta F_{\text{elect}}^{(1)}[q_i]} \equiv \langle e^{-\beta V[q_i]} \rangle_{U_0} \tag{2.18}$$

where $\langle\ \rangle_{U_0}$ and $V[q_i]$ are defined in (2.8) and (2.16) respectively.

However, the term $\int_0^1 ds\, v_{CW}(\mathbf{r}_1+\lambda_1\xi_1(s), \mathbf{r}_2+\lambda_2\xi_2(s))$, which is associated with the direct interaction between the two quantum particles by the Feynman–Kac representation (2.10) of the two-body matrix-element of the quantum Gibbs factor, cannot be reduced to an electrostatic energy such as $\int d\mathbf{r} \int d\mathbf{r}'\, q_1(\mathbf{r})\, v_{CW}(\mathbf{r}, \mathbf{r}')\, q_2(\mathbf{r}')$. There appears an extra purely quantum term, which is independent from the classical plasma,

$$\omega_{\alpha_1\alpha_2}[\mathbf{r}_1+\lambda_1\xi_1, \mathbf{r}_2+\lambda_2\xi_2] \equiv e_1 e_2 \int_0^1 ds_1 \int_0^1 ds_2 [\delta(s_1-s_2)-1] \times v_{CW}(\mathbf{r}_1+\lambda_1\xi_1(s_1), \mathbf{r}_2+\lambda_2\xi_2(s_2)) \tag{2.19}$$

The immersion free energy of the two quantum charges is expressed in terms of both the corresponding electrostatic free energy $F^{(2)}_{\text{elect}}[q_1, q_2]$ for two charge distributions $q_1(\mathbf{r})$ and $q_2(\mathbf{r})$ plus the quantum contribution $\omega_{\alpha_1\alpha_2}$,

$$e^{-\beta F^{(2)}_{\alpha_1\alpha_2}(\mathbf{r}_1, \mathbf{r}_2)} = \frac{1}{(2\pi\lambda_1^2)^{3/2}} \frac{1}{(2\pi\lambda_2^2)^{3/2}} \int D^0_W(\xi_1; x_1) \int D^0_W(\xi_2; x_2) \times \exp[-\beta\{F^{(2)}_{\text{elect}}[q_1, q_2] + \omega_{\alpha_1\alpha_2}[\mathbf{r}_1+\lambda_1\xi_1, \mathbf{r}_2+\lambda_2\xi_2]\}] \tag{2.20}$$

with

$$e^{-\beta F^{(2)}_{\text{elect}}[q_1, q_2]} = \exp\left[-\beta \int d\mathbf{r} \int d\mathbf{r}'\, q_1(\mathbf{r})\, v_{CW}(\mathbf{r}, \mathbf{r}')\, q_2(\mathbf{r}')\right] \times \langle \exp[-\beta\{V[q_1]+V[q_2]\}] \rangle_{U_0} \tag{2.21}$$

Eventually, from the previous definitions, the effective potential $\Phi^{\text{ext}}_{\alpha_1\alpha_2}(\mathbf{r}_1, \mathbf{r}_2)$ between two quantum particles introduced in (2.7) is related to an effective electrostatic potential

$$\Phi_{\text{elect}}[q_1, q_2] \equiv F^{(2)}_{\text{elect}}[q_1, q_2] - \sum_{i=1,2} F^{(1)}_{\text{elect}}[q_i] \tag{2.22}$$

between wires with random shapes plus a quantum contribution which is entirely independent from the classical plasma. Symmetry arguments imply that the immersion free energy $F^{(1)}_{\text{elect}}[q_i]$ depends only on the distance x_i from the wall and on the shape ξ_i. $\Phi_{\text{elect}}[q_1, q_2]$ appears if we introduce the measure with unit normalization, $\int \bar{D}_W(\xi_i; x_i) = 1$,

$$\bar{D}_W(\xi_i; x_i) \equiv \frac{e^{-\beta F^{(1)}_{\text{elect}}[q_i]}}{\int D^0_W(\xi_i; x_i)\, e^{-\beta F^{(1)}_{\text{elect}}[q_i]}} D^0_W(\xi_i; x_i) \tag{2.23}$$

According to (2.20) and (2.22) we get

$$\exp[-\beta\Phi^{\text{ext}}_{\alpha_1\alpha_2}(x_1, x_2; \mathbf{y})] - 1$$
$$= \int \bar{D}_W(\xi_1; x_1) \int \bar{D}_W(\xi_2; x_2)$$
$$\times \{\exp[-\beta\{\Phi_{\text{elect}}[q_1, q_2] + \omega_{\alpha_1\alpha_2}(\mathbf{r}_1 + \lambda_1\xi_1, \mathbf{r}_2 + \lambda_2\xi_2)\}] - 1\} \tag{2.24}$$

2.3. Origin of Algebraic Screening at Large Distances

According to known results about the classical regime for point particles, if ε_W is finite, $\Phi_{\text{elect}}[q_1, q_2]$ is expected to behave as $1/y^3$ when y goes to infinity, whereas, if ε_W is infinite, $\Phi_{\text{elect}}[q_1, q_2]$ decays faster than any inverse power law even along the wall.[5] The $1/y^3$ fall-off for ε_W finite is inferred from the following sum rule for point charges,[10]

$$\int_0^\infty dx \int_0^\infty dx' [\langle Q(\mathbf{r})\, Q(\mathbf{r}')\rangle_{U_0} - \langle Q(\mathbf{r})\rangle \langle Q(\mathbf{r}')\rangle_{U_0}]$$
$$\sim_{|\mathbf{y}_1 - \mathbf{y}_2| \to \infty} -\frac{\varepsilon_W}{8\pi^2\beta} \frac{1}{y^3} \tag{2.25}$$

with the same notations as in Eq. (1.3). Equation (2.25) is derived by use of the linear response theory from the assumption that an infinitesimal external static charge spread on the wall surface is completely screened by a classical plasma in the limit where the spatial variations of the external charge are infinitely smooth. This sum rule does not depend on the choice of the short-distance regularization v_{SR}, because it only arises from the long range of the potential. The $1/y^3$ behavior of $\Phi_{\text{elect}}[q_1, q_2]$ should remain valid after integration over the Brownian bridges ξ with the measure (2.23),

$$-\beta \int \bar{D}_W(\xi_1; x_1) \int \bar{D}_W(\xi_2; x_2)\, \Phi_{\text{elect}}[q_1, q_2]$$
$$\sim_{y \to \infty} \frac{f_{\text{elect}}(x_1, x_2; \varepsilon_W)}{y^3} \qquad \text{if } \varepsilon_W \text{ is finite} \tag{2.26}$$

Indeed, the Brownian bridge $\lambda_i \xi_i$ is confined to some finite area by the Gaussian weight involved in the Wiener measure $D^0_W(\xi_1; x_i)$, while $F^{(1)}_{\text{elect}}[q_i]$ vanishes far away from the distribution q_i. However, in the vicinity of an ideally conducting wall, the clustering is expected to be exponentially fast in all directions. In the following, we will recover that this is indeed the case in the weak-coupling limit.

Furthermore, for any kind of wall, the purely quantum term $\omega_{\alpha_1\alpha_2}$ also generates a $1/y^3$ tail when y goes to infinity, while x_1 and x_2 are kept fixed. On one hand, $\omega_{\alpha_1\alpha_2}$ decays at least as $1/r^3$ in all directions. The reason is that the property

$$\int_0^1 ds_1 \int_0^1 ds_2 [\delta(s_1 - s_2) - 1]\, f(s_1) = 0 \tag{2.27}$$

which is satisfied by any function f, ensures that in a large-distance Taylor expansion (namely for $y \gg x_1$, $y \gg x_2$ and $y \gg \lambda_i\, |\xi_i|$) only terms with at least one $\xi_1(s_1)$ *and* one $\xi_2(s_2)$ are not equal to zero. On the other hand, translation invariance is broken along the x axis so that $\int \bar{D}_W(\xi_i; x_i)[\xi_i]_x \neq 0$ whereas the corresponding averages in the directions parallel to the wall vanish. According to the value of $\partial^2 v_{CW}(x_1, x_2, \mathbf{y})/\partial x_1\, \partial x_2|_{x_1 = x_2 = 0}$, for any value of ε_W,

$$\begin{aligned} -\beta \int \bar{D}_W(\xi_1; x_1) \int \bar{D}_W(\xi_2; x_2)\, \omega_{\alpha_1\alpha_2}[\mathbf{r}_1 + \lambda_1 \xi_1, \mathbf{r}_2 + \lambda_2 \xi_2] \\ \sim_{y \to \infty} \frac{f_\omega(x_1, x_2; \varepsilon_W)}{y^3} \end{aligned} \tag{2.28}$$

with

$$f_\omega(x_1, x_2; \varepsilon_W) = -\frac{2\varepsilon_W}{1 + \varepsilon_W}\, \beta e_1 e_2 \lambda_1 \lambda_2 A\left(\frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2}\right) \tag{2.29}$$

and

$$\begin{aligned} A\left(\frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2}\right) = \int_0^1 ds_1 \int_0^1 ds_2 [\delta(s_1 - s_2) - 1] \int \bar{D}_W(\xi_1; x_1)[\xi_1(s_1)]_x \\ \times \int \bar{D}_W(\xi_2; x_2)[\xi_2(s_2)]_x \end{aligned} \tag{2.30}$$

The expression (2.29) is also valid in the limit where ε_W becomes infinite.

The conclusion of the previous general arguments is that $\Phi^{\text{ext}}_{\alpha_1\alpha_2}$ decays as $1/y^3$, whatever the electrical properties of the wall are, except that the tail is from purely quantum origin in the case $\varepsilon_W = \infty$. More precisely, the

$1/y^3$ tail may be decomposed into two contributions: according to (2.24), (2.26), and (2.28).

$$-\beta\Phi^{\text{ext}}_{\alpha_1\alpha_2}(x_1, x_2, \mathbf{y}) \sim_{y\to\infty} \frac{f_\omega(x_1, x_2; \varepsilon_W) + f_{\text{elect}}(x_1, x_2; \varepsilon_W)}{y^3} \tag{2.31}$$

where $f_{\text{elect}}(x_1, x_2; \varepsilon_W)$ contains the classical tail and its "diffraction" corrections due to quantum dynamics, while $f_\omega(x_1, x_2; \varepsilon_W)$ is proportional to $\hbar^2$ and vanishes in the strict classical limit.

In order to obtain more explicit results, we shall consider a particular regime where two parameters are small. First, a limit of weak coupling (1.10) between all charges will allow us to calculate Φ_{elect}. Afterwards, we will consider a subdomain of the weak-coupling regime where the quantum dynamical effects are not too strong in the sense that $\lambda_i \ll \xi_B$. Under this second assumption, the weight $\bar{D}_W(\boldsymbol{\xi}_i; x_i)$, which involves the electrostatic immersion free energy of $q_i(\mathbf{r})$, is reduced to the normalized measure $\bar{D}^0_W(\boldsymbol{\xi}_i; x_i)$ for free particles, and the averages giving $f_{\text{elect}}(x_1, x_2; \varepsilon_W)$ and $f_\omega(x_1, x_2; \varepsilon_W)$ can be calculated up to second order in the parameter λ_i/ξ_B in some special regime.

3. WEAK-COUPLING LIMIT

For the sake of simplicity, the classical plasma is chosen to be a One-component Plasma in the following. (All results are also true for a multi-component plasma and are merely written with a larger number of indexes.) Every moving particle carries a charge e. At equilibrium, the particle density in the bulk is denoted by ρ. According to the local neutrality relation $\langle Q_{\text{bulk}}(\mathbf{r})\rangle_{U_0} = 0$, which is always satisfied far from the wall,[11] the charge density of the rigid background is equal to $-e\rho$. For a given density ρ, the difference between the classical OCP and an ideal gas is entirely characterized by the coupling constant

$$\Gamma \equiv \frac{\beta e^2}{a} \propto \left(\frac{a}{\xi_B}\right)^2 \tag{3.1}$$

where a and ξ_B are defined in Eqs. (1.10) and (1.11), so that $\xi_B = 1/\sqrt{4\pi\beta\rho e^2}$.

3.1. Electrostatic Energies in the Debye Self-Consistent Approximation

We just sum up the mainlines of the calculation while some details are given in Appendix A. By use of an expansion in q_i up to second order, the immersion free energies defined in Eqs. (2.18) and (2.21) are expressed in terms of both the charge density

$$\langle Q(\mathbf{r})\rangle_{U_0} = e[\rho(x) - \rho] \tag{3.2}$$

and the charge-charge correlation function $\langle Q(\mathbf{r})\, Q(\mathbf{r}')\rangle_{U_0} - \langle Q(\mathbf{r})\rangle_{U_0} \langle Q(\mathbf{r}')\rangle_{U_0}$ in the classical plasma in the absence of the two quantum charges. In the vicinity of any wall the total density of moving species in a multicomponent plasma is not uniform. Indeed, on one hand, the force per unit surface exerted by the wall on the fluid is opposite to the kinetic pressure $\rho(x=0)\, k_B T$ determined by the total density $\rho(x=0)$ of moving particles on the wall. On the other hand, this force must be balanced by the bulk pressure on the other side of a fluid slab between the plane $x=0$ and a parallel plane at x very large. Since the bulk pressure in the presence of interactions is not equal to its value $\rho k_B T$ in an ideal gas, $\rho(x=0) \neq \rho$ and there must be a nonuniform density profile $\rho(x)$. The result is also valid in the case of the OCP, though there is an extra electrostatic contribution from the background to the thermodynamical bulk pressure, because the latter is defined in such a way that the system remains neutral when its volume is varied.[12] Thus, it is convenient to express the charge-charge correlation function for the classical plasma in terms of the Ursell function $h_{\mathrm{cl},\,W}(\mathbf{r}, \mathbf{r}')$,

$$\begin{aligned}\langle Q(\mathbf{r})\, Q(\mathbf{r}')\rangle_{U_0} &- \langle Q(\mathbf{r})\rangle_{U_0} \langle Q(\mathbf{r}')\rangle_{U_0} \\ &= e^2[\rho(x)\, \rho(x')\, h_{\mathrm{cl},\,W}(\mathbf{r}, \mathbf{r}') + \rho(x)\, \delta(\mathbf{r} - \mathbf{r}')]\end{aligned} \tag{3.3}$$

The mean-field approximation may be introduced by the same scheme as in the bulk. Let $e\phi_W(\mathbf{r}, \mathbf{r}')$ be the total potential created at $\mathbf{r}'$ by a charge e located at $\mathbf{r}$ and by its polarization cloud. The charge density of this cloud at $\mathbf{r}''$ is the excess charge $eh_{\mathrm{cl},\,W}(\mathbf{r}, \mathbf{r}'')\, \rho(x'')$ with respect to the mean charge density $e[\rho(x'') - \rho]$ and

$$e\phi_W(\mathbf{r}, \mathbf{r}') \equiv ev_{CW}(\mathbf{r}, \mathbf{r}') + \int d\mathbf{r}''\, h_{\mathrm{cl},\,W}(\mathbf{r}, \mathbf{r}'')\, e\rho(x'')\, v_{CW}(\mathbf{r}'', \mathbf{r}') \tag{3.4}$$

[v_{SR} is not involved in ϕ_W, because in the following the latter is only aimed to describe correlations at length scales that are larger than the mean interparticle distance a; however, v_{SR} should be included in the description of thermodynamic quantities such as the profile density $\rho(x)$ (see Section 2.1.)]. The self-consistent mean-field approximation amounts to replacing $h_{\mathrm{cl},\,W}$ in Eq. (3.4) by

$$h_{\mathrm{cl},\,MFW}(\mathbf{r}, \mathbf{r}') \equiv -\beta e^2 \phi_{MFW}(\mathbf{r}, \mathbf{r}') \tag{3.5}$$

Thus the mean-field total potential $\phi_{MFW}(\mathbf{r}, \mathbf{r}')$ is the solution of the integral equation derived from the combination of Eqs. (3.4) and (3.5),

$$\phi_{MFW}(\mathbf{r}, \mathbf{r}') = v_{CW}(\mathbf{r}, \mathbf{r}') - \beta e^2 \int d\mathbf{r}'' \, \rho(x'') \, \phi_{MFW}(\mathbf{r}, \mathbf{r}'') \, v_{CW}(\mathbf{r}'', \mathbf{r}') \tag{3.6}$$

The combination of Eq. (3.6) with the Poisson Eq. (1.1) satisfied by v_{CW} (with the corresponding boundary conditions) shows that $\phi_{MFW}(\mathbf{r}, \mathbf{r}')$ in the case of the Coulomb potential is the solution of the linearized Poisson–Boltzmann equation

$$\Delta_{r'} \phi_{MFW}(\mathbf{r}, \mathbf{r}') - \kappa^2(x') \, \phi_{MFW}(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}') \qquad \text{for} \quad x' > 0 \tag{3.7}$$

with

$$\kappa(x') \equiv \sqrt{4\pi\beta\rho(x') \, e^2} \tag{3.8}$$

ϕ_{MFW} obeys the same boundary conditions as v_{CW}, because it is also an electrostatic potential. Since $v_{CW}(\mathbf{r}, \mathbf{r}') = v_{CW}(\mathbf{r}', \mathbf{r})$, according to Appendix A, the Coulomb mean-field free energy takes the general form

$$\begin{aligned} F^{(1)}_{\text{elect}, \, W}[q_i] = \int d\mathbf{r} \int d\mathbf{r}' \, q_i(\mathbf{r}) \, v_{CW}(\mathbf{r}, \mathbf{r}') \, e[\rho(x') - \rho] \\ + F^{(1)}_{\text{elect}, \, MFW}[q_i] + O(q_i^3) \end{aligned} \tag{3.9}$$

where $O(q_i^3)$ denotes a term of order q_i^3 and the contribution of order q_i^2 reads

$$F^{(1)}_{\text{elect}, \, MFW}[q_i] \equiv \frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' \, q_i(\mathbf{r}) [\phi_{MFW}(\mathbf{r}, \mathbf{r}') - v_{CW}(\mathbf{r}, \mathbf{r}')] \, q_i(\mathbf{r}') \tag{3.10}$$

At the lowest order in q, the mean-field value of $\Phi_{\text{elect}}[q_1, q_2]$ defined in Eq. (2.22) is

$$\Phi_{\text{elect}, \, MFW}[q_1, q_2] = \int d\mathbf{r} \int d\mathbf{r}' \, q_1(\mathbf{r}) \, \phi_{MFW}(\mathbf{r}, \mathbf{r}') \, q_2(\mathbf{r}') \tag{3.11}$$

An extra assumption can be performed in the weak-coupling limit $\Gamma \ll 1$ of the coulombic mean-field approximation:[13] the profile density $\rho(x)$ is replaced by its uniform bulk value ρ in the whole half-space occupied by the plasma. The corresponding expression of ϕ_{MFW} will be called the Debye value ϕ_{DW} in the following. ϕ_{DW} is the exact value of

ϕ_{MFW} at leading order in Γ and the correction to ϕ_{DW} arising from the nonuniformity of the profile density is of order $\Gamma^{3/2} \times [\phi_{DW}(\mathbf{r}, \mathbf{r}') - v_{CW}(\mathbf{r}, \mathbf{r}')]$. Indeed, according to Eq. (3.6), $\phi_{MFW} - \phi_{DW}$ is of order $-\beta e^2 \int dx'' [\rho(x'') - \rho]\, \phi_{DW}(\mathbf{r}, \mathbf{r}'')\, v_{CW}(\mathbf{r}'', \mathbf{r})$ where ϕ_{DW} satisfies Eq. (3.6) with $\rho(x'')$ replaced by ρ, while the relative correction $[\rho(x) - \rho]/\rho$ derived from ϕ_{DW} through the BGY equation proves to be only of order $\Gamma^{3/2}$,[13]

$$\rho(x) - \rho = \rho \times O(\Gamma^{3/2}) \tag{3.12}$$

(The latter result can also be readily inferred from the balance between the kinetic energy and the bulk pressure, whose Debye value differs from $\rho k_B T$ only through a term proportional to $(1/\xi_B)^3 \propto \rho \Gamma^{3/2}$ (see Eq. (3.1)).) Half integer-powers in the coupling constant Γ arise from screening collective effects.

At the lowest order in Γ, only the second term in the r.h.s. of Eq. (3.9) with ϕ_{MFW} replaced by ϕ_{DW} contributes to the immersion free energy. Indeed, Eq. (3.9) arises from Eq. (A1) and, according to Eq. (A4), $\langle Q_D^{\text{ind}}(\mathbf{r}) \rangle_{U_0} = q_i \times O(\rho\Gamma)$ whereas, according to Eq. (3.12), the charge density associated with $\rho(x) - \rho$ by Eq. (3.2) is $\langle Q(x) \rangle_{U_0} = e \times O(\rho \Gamma^{3/2})$. As a consequence

$$F^{(1)}_{\text{elect},\, W}[q_i] = F^{(1)}_{\text{elect},\, DW}[q_i] \times [1 + O(\Gamma^{1/2})] \tag{3.13}$$

where the term of order $\Gamma^{1/2}$ arises from coupling effects, in particular from the correction to the homogeneous density profile that are induced by Coulomb interactions. Besides, since the relative correction to ϕ_{DW} arising from the nonuniformity of the profile density is of order $\Gamma^{3/2}$ (see previous paragraph), the corresponding correction to $\Phi_{\text{elect},\, MFW}[q_1, q_2] - \Phi_{\text{elect},\, DW}[q_1, q_2]$ is of relative order $O(\Gamma^{3/2})$. However, we do not know the order of the corrections which go beyond the mean-field approximation and we can only state that

$$\Phi_{\text{elect},\, W}[q_1, q_2] = \Phi_{\text{elect},\, DW}[q_1, q_2] \times [1 + O(\Gamma^{1/2})] \tag{3.14}$$

The Debye potential ϕ_{DW} can be explicitly calculated as the solution of Eq. (3.7) where $\kappa(x')$ is replaced by $\kappa = \sqrt{4\pi\beta\rho e^2} = 1/\xi_B$. As in the case of v_{CW} (see Eq. (2.1)), ϕ_{DW} may be written in a single form for any value of the dielectric constant ε_W of the wall.[14] ϕ_{DW} appears as the sum of the solution $\phi_{D\text{bulk}}$ in the bulk

$$\phi_{D\text{bulk}}(|\mathbf{r}_1 - \mathbf{r}_2|) = \frac{e^{-\kappa |\mathbf{r}_1 - \mathbf{r}_2|}}{|\mathbf{r}_1 - \mathbf{r}_2|} \tag{3.15}$$

plus a correction which, by itself, takes into account the presence of the wall. Indeed, because of translational invariance in the direction $\mathbf{y}$ parallel to the wall, a Fourier transform in this direction allows one to reduce Eq. (3.7) with second order partial derivatives to a one-dimensional equation with second order derivatives. Since the bulk solution $\phi_{D\text{bulk}}$ is a particular solution of the equation with the delta distribution, the solution of the corresponding homogeneous equation that may be added to $\phi_{D\text{bulk}}$ is entirely determined by the boundary conditions (far away from the wall and at the interface between the fluid and the wall). The explicit solution of Eq. (3.7) depends on which side of the wall $\mathbf{r}$ and $\mathbf{r}'$ are. In the domain of interest, $x > 0$ and $x' > 0$,

$$\phi_{DW}(x, x', |\mathbf{y}|) = \phi_{D\text{bulk}}(|\mathbf{r} - \mathbf{r}'|) + \int \frac{d^2\mathbf{k}_{||}}{2\pi} e^{-i\mathbf{k}_{||}\cdot\mathbf{y}} \frac{1}{\sqrt{\kappa^2 + \mathbf{k}_{||}^2}} \frac{\sqrt{\kappa^2 + \mathbf{k}_{||}^2} - \varepsilon_W |\mathbf{k}_{||}|}{\sqrt{\kappa^2 + \mathbf{k}_{||}^2} + \varepsilon_W |\mathbf{k}_{||}|} \times \exp[-\sqrt{\kappa^2 + \mathbf{k}_{||}^2}(x_1 + x_2)] \tag{3.16}$$

Only the amplitude of the Fourier transform of $\phi_{DW} - \phi_{D\text{bulk}}$ depends on the electrostatic properties of the wall. As in the case of $v_{CW} - v_{C\text{bulk}}$, the difference $\phi_{DW} - \phi_{D\text{bulk}}$ might also be interpreted in terms of image-charge distributions, each of which would be characterized by a wave vector, because the Fourier transform of Eq. (3.15) restricted to the components of $\mathbf{r}$ which are parallel to the wall reads

$$\phi_{D\text{bulk}}(x, x', \mathbf{k}_{||}) = 2\pi \frac{\exp[-\sqrt{\kappa^2 + \mathbf{k}_{||}^2}\, |x - x'|]}{\sqrt{\kappa^2 + \mathbf{k}_{||}^2}} \tag{3.17}$$

We notice that, under the assumption that the external charge e_i is of order e, according to Eqs. (3.1), (3.10), and (3.16)

$$-\beta F^{(1)}_{\text{elect}, DW}[q_i] = O(\Gamma^{3/2}) \tag{3.18}$$

Besides, the integrals of the exponential of $F^{(1)}_{\text{elect}, DW}[q_i]$ and $\Phi_{\text{elect}, DW}[q_1, q_2]$ with the measure $D^0_W(\xi; x)$ are finite, in spite of the existence of attraction between plasma charges and images inside the wall, because the measure $D^0_W(\xi; x)$ smooths the corresponding short-distance singularities. Thus, the quantum free energies $F^{(1)}_{\alpha_i, DW}(\mathbf{r}_1)$ and $F^{(2)}_{\alpha_1\alpha_2, DW}(\mathbf{r}_1, \mathbf{r}_2)$, defined in Eqs. (2.17) and (2.20) and calculated in the Debye approximation, are finite.

3.2. Algebraic Electrostatic Tails

For a dielectric or a plain wall, the small-$|\mathbf{k}_{||}|$ expansion of the integrand of $d^2\mathbf{k}_{||}/(2\pi)^2$ in Eq. (3.16) contains nonanalytic terms in the components of $\mathbf{k}_{||}$ and the singularity at lowest order in $|\mathbf{k}_{||}|$ reads

$$-4\pi\,|\mathbf{k}_{||}|\times\varepsilon_W\frac{e^{-\kappa(x_1+x_2)}}{\kappa^2} \tag{3.19}$$

Subsequently, according to the theory of distributions (see p. 363 in ref. 15),

$$\phi_{DW}(x_1,x_2,|\mathbf{y}|)\sim_{y\to\infty}\frac{2}{y^3}\varepsilon_W\frac{e^{-\kappa(x_1+x_2)}}{\kappa^2} \tag{3.20}$$

We notice that Eq. (3.20) implies that the approximated charge correlation function given by Eq. (3.3) with $h_{\text{cl},\,DW}$ in place of $h_{\text{cl},\,W}$ happens to obey the exact classical sum rule (2.25). In the weak-coupling limit, according to Eq. (3.13), $F^{(1)}_{\text{elect}}[q_i]$ tends to $F^{(1)}_{\text{elect},\,DW}[q_i]$ where $F^{(1)}_{\text{elect},\,DW}[q_i]$ is derived from Eqs. (3.10) and (3.16) and depends only on x_i and ξ_i. In the same way, Φ_{elect} tends to $\Phi_{\text{elect},\,DW}$ given by Eqs. (3.11) and (3.16) and the coefficient of he corresponding $1/y^3$ tail (2.26) tends to $f_{\text{elect},\,DW}(x_1,x_2;\varepsilon_W)$,

$$f_{\text{elect}}(x_1,x_2;\varepsilon_W)=f_{\text{elect},\,DW}(x_1,x_2;\varepsilon_W)\times\{1+O(\Gamma^{1/2})\} \tag{3.21}$$

According to Eqs. (2.15) and (3.20),

$$\begin{aligned} f_{\text{elect},\,DW}(x_1,x_2;\varepsilon_W)= &-\beta e_1e_2\frac{2\varepsilon_W}{\kappa^2}\int_0^1 ds_1\int_0^1 ds_2\\ &\times\int\bar{D}_{DW}(\xi_1;x_1)\int\bar{D}_{DW}(\xi_2;x_2)\\ &\times\exp\{-\kappa(x_1+\lambda_1[\xi_1(s_1)]_x+x_2+\lambda_2[\xi_2(s_2)]_x)\} \end{aligned} \tag{3.22}$$

In Eq. (3.22) the measure in the weak coupling limit, $\bar{D}_{DW}$, is defined by Eq. (2.23) with $F^{(1)}_{\text{elect}}[q_i]$ replaced by its Debye approximation (3.10). (We notice that, according to Eq. (3.16), when ε_W is finite and $\varepsilon_W\neq 1$, $F^{(1)}_{\text{elect},\,DW}[q_i]$ tends to its bulk value only algebraically, as $1/x_i$, when x_i goes to infinity, whereas it decays exponentially fast when $\varepsilon_W=1$.)

Contrarily, in the case of an ideally conducting wall, according to Eq. (3.17), the expression (3.16) gives

$$\phi_{DW}(x_1,x_2,|\mathbf{y}|)|_{\varepsilon_W=\infty}=\phi_{D\text{bulk}}(|\mathbf{r}_1-\mathbf{r}_2|)-\phi_{D\text{bulk}}(|\mathbf{r}_1-\mathbf{r}_2^*|) \tag{3.23}$$

for $x_1 > 0$ and $x_2 > 0$. (The latter result might also be directly retrieved from the image method for a potential that vanishes in the wall). Then $\phi_{DW}(x_1, x_2, |\mathbf{y}|)$ decreases exponentially fast in all directions in the plasma, in agreement with the general results recalled in Section 2.3. (We notice that the immersion free energy $F^{(1)}_{\text{elect}, DW}[q_i]$, tends to its bulk value exponentially fast when x_i goes to infinity.) According to Eq. (3.11) and symmetry arguments,

$$-\beta \int \bar{D}_{DW}(\xi_1; x_1) \int \bar{D}_{DW}(\xi_2; x_2)\, \Phi_{\text{elect}, DW}[q_1, q_2]|_{\varepsilon_W = \infty}$$

$$\sim_{y \to \infty} -2\beta\kappa e_1 e_2 \left[x_1 + \lambda_1 \int_0^1 ds \int \bar{D}_{DW}(\xi_1; x_1)[\xi_1]_x(s) \right]$$

$$\times \left[x_2 + \lambda_2 \int_0^1 ds \int \bar{D}_{DW}(\xi_2; x_2)[\xi_2]_x(s) \right] \times \frac{e^{-\kappa y}}{y^2} \tag{3.24}$$

Eventually, no algebraic tail arises from $\Phi_{\text{elect}, DW}$ when the wall is an ideal conductor.

4. WEAKLY-DELOCALIZED QUANTUM CHARGE IN THE WEAK-COUPLING LIMIT

4.1. More Explicit Path-Integral Measure

In the regime where $\kappa\lambda_i \ll 1$ the electrostatic immersion free energy $F^{(1)}_{\text{elect}, W}[q_i]$ of the distribution $q_i(\mathbf{r})$ with shape $\lambda_i \xi_i(s)$ can be expanded around its value $F^{(1)}_{\text{cl}, W}(e_i; x_i)$ for a classical point particle whose charge e_i is equal to the total charge of the distribution $q_i(\mathbf{r})$. $\beta F^{(1)}_{\text{cl}, W}(e_i; x_i)$ is equal to the expression of $\beta F^{(1)}_{\text{elect}, W}[q_i]$ when $q_i(\mathbf{r})$ is replaced by $e_i \delta(\mathbf{r} - \mathbf{r}_i)$. According to Eq. (3.13), the free energies in the weak-coupling limit are equal to their Debye approximations up to a relative correction of order $\Gamma^{1/2}$. Thus,

$$\beta F^{(1)}_{\text{elect}, W}[q_i] = \beta F^{(1)}_{\text{cl}, W}(e_i; x_i) + [\beta F^{(1)}_{\text{elect}, DW}[q_i] - \beta F^{(1)}_{\text{cl}, DW}(e_i; x_i)]$$
$$\times [1 + O(\Gamma^{1/2})] \tag{4.1}$$

According to Eq. (3.18), the Debye approximation itself is of order $\beta\kappa e^2 = O(\Gamma^{3/2})$, and the scaling change $\mathbf{k}_{//} = \kappa \mathbf{u}$ shows that the expansion of

$F^{(1)}_{\text{elect}, DW}[q_i]$ with respect to the dimensionless parameter $\kappa\lambda_i$ takes the form,

$$\beta F^{(1)}_{\text{elect}, DW}[q_i] = \beta F^{(1)}_{\text{cl}, DW}(e_i; x_i) + \Gamma^{3/2} \times \left\{ \sum_{n=1}^{\infty} (\kappa\lambda_i)^n \, C_n[\xi_i; \kappa x_i] \right\} \tag{4.2}$$

Subsequently, according to (4.1),

$$\beta F^{(1)}_{\text{elect}, W}[q_i] = \beta F^{(1)}_{\text{cl}, W}(e_i; x_i) + O(\Gamma^{3/2}\kappa\lambda_i) \tag{4.3}$$

Since the classical free energies do not depend on the shape ξ_i by construction of the $\kappa\lambda_i$-expansion, the normalized measures $\bar{D}_W(\xi; x)$ and $\bar{D}_{DW}(\xi; x)$ tend to the normalized free value $\bar{D}^0_W(\xi; x)$ when $\kappa\lambda_i$ vanishes,

$$\bar{D}_{DW}(\xi; x) = \bar{D}^0_W(\xi; x)[1 + O(\Gamma^{3/2}(\kappa\lambda_i))] \tag{4.4}$$

as well as

$$\bar{D}_W(\xi; x) = \bar{D}^0_W(\xi; x)[1 + O(\Gamma^{3/2}(\kappa\lambda_i))] \tag{4.5}$$

with

$$\bar{D}^0_W(\xi; x) = \frac{D^0_W(\xi; x)}{1 - \exp[-2(x/\lambda)^2]} \tag{4.6}$$

according to (2.14).

Subsequently, some path integrals can be performed explicitly. This is the case for the first and second moments of the free measure $\bar{D}^0_W(\xi; x)$ as well as for their integrals over the parameter s. Let us introduce the notations

$$\bar{\mu}^{(1)}(x_i/\lambda_i) \equiv \int_0^1 ds \int \bar{D}^0_W(\xi_i; x_i)[\xi_i(s)]_x \tag{4.7}$$

and

$$\bar{\mu}^{(2)}(x_i/\lambda_i) \equiv \int_0^1 ds \int \bar{D}^0_W(\xi_i; x_i)([\xi_i(s)]_x)^2 \tag{4.8}$$

where the dependence upon λ_i arises from the measure $\bar{D}^0_W(\xi_i; x_i)$. For the sake of conciseness, we set $\tilde{x} \equiv x/\lambda$. According to Appendix B,

$$\bar{\mu}^{(1)}(\tilde{x}) = \sqrt{\frac{\pi}{2}} \frac{\tilde{x}^2 \, \text{Erfc}(\sqrt{2}\,\tilde{x})}{[1 - \exp(-2\tilde{x}^2)]} \tag{4.9}$$

and

$$\bar{\mu}^{(2)}(\tilde{x}) = \frac{1}{6} + \frac{(2/3)\,\tilde{x}^2 e^{-2\tilde{x}^2} - \sqrt{2\pi}\,\tilde{x}^3\,\mathrm{Erfc}(\sqrt{2}\,\tilde{x})}{[1 - \exp(-2\tilde{x}^2)]} \tag{4.10}$$

where the complementary error function is defined as

$$\mathrm{Erfc}(u) \equiv 1 - \frac{2}{\sqrt{\pi}} \int_0^u dt\, \exp[-t^2] \tag{4.11}$$

These expressions for $\bar{\mu}^{(1)}(\tilde{x})$ and $\bar{\mu}^{(2)}(\tilde{x})$ do tend to their respective bulk values 0 and 1/6 when $\tilde{x}$ goes to infinity. Besides they take their maximum values at the origin: $\bar{\mu}^{(1)}(\tilde{x}=0) = \sqrt{\pi}/(2\sqrt{2})$ and $\bar{\mu}^{(2)}(\tilde{x}=0) = 1/2$.

Eventually, we get explicit formulas for the $1/y^3$ tails in the double expansion $\Gamma \ll 1$ and $\kappa\lambda_i \ll 1$. According to (4.5), the purely quantum term reads $f_\omega = f_\omega^*[1 + O(\Gamma^{3/2}\kappa\lambda_i)]$, where f_ω^* is given by (2.29) with $\bar{D}_W^0(\xi; x)$ in place of $\bar{D}_W(\xi; x)$. The result is

$$f_\omega^*(x_1, x_2; \varepsilon_W) = -\beta e_1 e_2 \frac{2\varepsilon_W}{1+\varepsilon_W} \lambda_1 \lambda_2 A^*(\tilde{x}_1, \tilde{x}_2) \tag{4.12}$$

with $A^*(\tilde{x}_1, \tilde{x}_2) = a(\tilde{x}_1; \tilde{x}_2) + a(\tilde{x}_2; \tilde{x}_1)$ and

$$\begin{aligned} a(\tilde{x}_1; \tilde{x}_2) = \Bigg\{ & \mathrm{Erfc}(\sqrt{2}\,\tilde{x}_1) \left[\left(\frac{1}{6} + \sqrt{\frac{\pi}{2}}\,\tilde{x}_1 + \frac{2}{3}\tilde{x}_1^2 \right) e^{-2\tilde{x}_2^2} - \frac{\pi}{4}\tilde{x}_1\tilde{x}_2\,\mathrm{Erfc}(\sqrt{2}\,\tilde{x}_2) \right] \\ & - \frac{1}{3} e^{-2(\tilde{x}_1^2 + \tilde{x}_2^2)} \left(\sqrt{\frac{2}{\pi}}\,\tilde{x}_1 + 1 \right) \\ & + \sqrt{\frac{2}{\pi}}\,\tilde{x}_1 \int_1^\infty du\, \frac{\sqrt{u^2-1}}{u} e^{-2\tilde{x}_1^2 u^2}\,\mathrm{Erfc}(\sqrt{2}\,\tilde{x}_2 u) \\ & - \frac{1}{3}\sqrt{\frac{2}{\pi}}\,\tilde{x}_1 e^{-2(\tilde{x}_1^2+\tilde{x}_2^2)} \int_0^\infty du\, \frac{u^3}{(u^2+1)^{3/2}} e^{-2\tilde{x}_1^2 u^2}\,\mathrm{Erfc}(\sqrt{2}\,\tilde{x}_2 u) \Bigg\} \\ & \times \tilde{x}_1 \tilde{x}_2 \frac{1}{[1-\exp(-2\tilde{x}_1^2)][1-\exp(-2\tilde{x}_2^2)]} \end{aligned} \tag{4.13}$$

We notice that $f_\omega^*(x_1, x_2; \varepsilon_W)$ is independent from the density. This property is linked to the fact that $\omega_{\alpha_1\alpha_2}$ measures the difference between the electrostatic potential and the potential given by the Feynman–Kac formula (2.11) even in the absence of classical plasma and f_ω^* is a zero-density limit in the sense that $\lambda_i \ll \xi_B$. The small-$\tilde{x}_1$ expansion of $A^*(\tilde{x}_1, \tilde{x}_2)$ involves only even powers of $\tilde{x}_1$. When both $\tilde{x}_1$ and $\tilde{x}_2$ tend to zero, $A^*(\tilde{x}_1, \tilde{x}_2)$ tends to a finite constant. For a fixed $\tilde{x}_2$

$$A^*(\tilde{x}_1, \tilde{x}_2) \sim_{\tilde{x}_1 \to \infty} \frac{1}{12}\tilde{x}_1 e^{-2\tilde{x}_1^2}\tilde{x}_2 \alpha(\tilde{x}_2) \tag{4.14}$$

where $\alpha(\tilde{x}_2) \sim \exp(-2\tilde{x}_2^2)$ when $\tilde{x}_2$ is also sent to infinity, while $\tilde{x}_2\alpha(\tilde{x}_2)$ remains finite if $\tilde{x}_2$ vanishes.

4.2. More Restricted Regime

According to Eq. (3.21), the tail f_{elect} is calculated only at the leading order in Γ, where it is equal to $f_{\text{elect}, DW}$, with a relative correction of order $\Gamma^{1/2}$. Moreover, according to (4.4), the $\kappa\lambda_i$-expansions of $f_{\text{elect}, DW}$ derived from Eq. (3.22) must be made up to an order lower than $\Gamma^{3/2}\kappa\lambda_i$ (for $i = 1, 2$) which is the order of the relative correction that is neglected in the expression (4.4) of $\bar{D}_{DW}$. Under the extra assumption that $\Gamma^{1/2} \ll (\kappa\lambda_i)^2 \ll 1$, which implies that $\Gamma^{3/2} \ll \kappa\lambda_i$, the $\kappa\lambda_i$-expansions of $f_{\text{elect}, DW}$ up to the second order included give the expansion of the exact value of f_{elect} up to a term $o(\kappa^2\lambda_i^2)$ of order higher than $\kappa^2\lambda_i^2$,

$$f_{\text{elect}} = f^*_{\text{elect}, DW}[1 + o(\kappa^2\lambda_i^2)] \tag{4.15}$$

The three conditions $\Gamma \ll 1$, $\kappa\lambda_i \ll 1$ and $\Gamma^{1/2} \ll (\kappa\lambda_i)^2$ correspond to the hierarchy of length scales

$$\frac{a}{\xi_B} \ll \left(\frac{\lambda_i}{\xi_B}\right)^2 \ll 1 \tag{4.16}$$

This implies that $a \ll \lambda_i \ll \xi_B$. In (4.15)

$$f^*_{\text{elect}, DW}(x_1, x_2; \varepsilon_W) = f_{\text{cl}, DW}(x_1, x_2; \varepsilon_W) + f^*_{\text{diff}, DW}(x_1, x_2; \varepsilon_W) \tag{4.17}$$

$f_{\text{cl}, DW}/y^3$, which coincides with the value (3.22) of $f_{\text{elect}, DW}/y^3$ when $\lambda_i = 0$, is just the classical $1/y^3$ tail in the Debye approximation,

$$f_{\text{cl}, DW}(x_1, x_2; \varepsilon_W) = -\beta e_1 e_2 \frac{2\varepsilon_W}{\kappa^2} e^{-\kappa(x_1 + x_2)} \tag{4.18}$$

The so-called diffraction correction $f^*_{\text{diff}, DW}$ coincides with the sum of the first two terms in the $\kappa\lambda_i$-expansion of Eq. (3.22) when $\bar{D}_{DW}(\xi; x)$ is replaced by $\bar{D}^0_W(\xi; x)$,

$$f^{*}_{\text{diff},\,DW}(x_1, x_2; \varepsilon_W)$$
$$= f_{\text{cl},\,DW}(x_1, x_2; \varepsilon_W)$$
$$\times \Big\{ -\kappa\lambda_1 \bar{\mu}^{(1)}(\tilde{x}_1) - \kappa\lambda_2 \bar{\mu}^{(1)}(\tilde{x}_2) \tag{4.19a}$$
$$+\kappa^2 \left[\frac{\lambda_1^2}{2} \bar{\mu}^{(2)}(\tilde{x}_1) + \lambda_1\lambda_2 \bar{\mu}^{(1)}(\tilde{x}_1)\, \bar{\mu}^{(1)}(\tilde{x}_2) + \frac{\lambda_2^2}{2} \bar{\mu}^{(2)}(\tilde{x}_2) \right] \Big\} \tag{4.19b}$$

The signs of these corrections depend on the order in $\kappa\lambda_i$.

4.3. Comparison of the Tails from Various Origins

In order to compare the amplitudes of the various tails, we consider their values for $\lambda_1 = \lambda_2 \equiv \lambda$ and for $x_1 = x_2 = x$. Let us denote the maximum value of $f_{\text{cl},\,DW}$ by $\max_{\text{cl},\,DW} \equiv -\beta e_1 e_2 2\varepsilon_W/\kappa^2$. For any finite value of ε_W, we consider the ratios

$$\frac{f^{*}_{\text{elect},\,DW}(x, x; \varepsilon_W)}{\max_{\text{cl},\,DW}}$$
$$= e^{-2\kappa x}\{1 - 2\kappa\lambda\bar{\mu}^{(1)}(\tilde{x}) + (\kappa\lambda^2)[\bar{\mu}^{(2)}(\tilde{x}) + (\bar{\mu}^{(1)}(\tilde{x}))^2]\} \tag{4.20}$$

and

$$\frac{f^{*}_{\omega}(x, x; \varepsilon_W)}{\max_{\text{cl},\,DW}} = (\kappa\lambda)^2 \frac{1}{1+\varepsilon_W} A^{*}(\tilde{x}, \tilde{x}) \tag{4.21}$$

where the expressions in (4.20) and (4.21) are derived from (4.12) and (4.17). At the considered order in our double expansion with respect to Γ and $\kappa\lambda$ with $\Gamma^{1/2} \ll (\kappa\lambda)^2$, the ratio (4.21) depends only on $\kappa\lambda$ (and not on Γ).

Since $\kappa\lambda \ll 1$, the total coefficient $f^{*}(x, x; \varepsilon_W) = f^{*}_{\text{elect},\,DW}(x, x; \varepsilon_W) + f^{*}_{\omega}(x, x; \varepsilon_W)$ nearly coincides with its classical value $f_{\text{cl},\,DW}(x, x; \varepsilon_W)$ at distances larger than a few de Broglie wavelengths λ and it falls of exponentially fast over the distance $\xi_B = 1/\kappa$. All quantum corrections are concentrated near the wall over a range of order $\lambda \ll \xi_B$, as shown in Fig. 1, where $\kappa\lambda = 0.1$ is yet not so small.

The first-order correction in $\hbar$ arises only from the diffraction correction to the classical coefficient. It lowers the value of $f^{*}(x, x; \varepsilon_W)$ near the wall. This diffraction correction, of order $\kappa\lambda$, arises from the nonvanishing

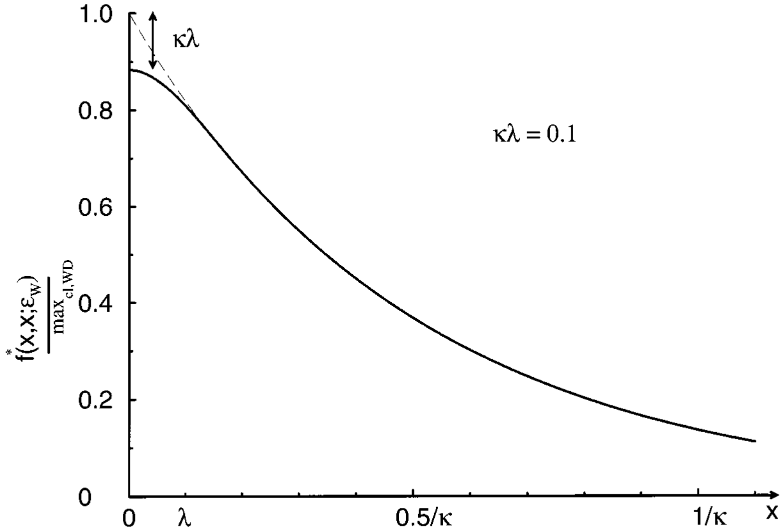

Fig. 1. The profile of the global coefficient $f^*(x, x; \varepsilon_W)/\max_{\mathrm{cl},\,WD}$ of the $1/y^3$ tail of the correlation between the two quantum particles in the case $x_1 = x_2 = x$ and in the regime (4.16) for $\kappa\lambda = 0.1$ and any finite value of ε_W. The slope at the origin does not vanish, contrarily to what might be inferred from the figure at the chosen scales.

value of the first moment of ξ and tends to zero in the bulk where rotational symmetry is restored, as it can be checked by inspection of Eqs. (4.20) and (4.9). On the contrary, the diffraction correction of relative order $(\kappa\lambda)^2$ involves the second moment of ξ, which does not vanish far away from the wall. Thus, at large distances, the $(\kappa\lambda)^2$-term in $f^*_{\mathrm{diff},\,DW}$ tends to $(\kappa\lambda)^2/6$ times $f_{\mathrm{cl},\,DW}$.

The purely quantum tail $f^*_{\omega}(x, x; \varepsilon_W)$ is completely negligible when ε_W is finite for two reasons. First $f^*_{\omega}(x, x; \varepsilon_W)$ is of relative order $(\kappa\lambda)^2$ with respect to the classical tail. Second, its amplitude is 10^2 times smaller than that of the diffraction correction at the same order in $\hbar$. In Fig. 2 a logarithmic scale is used to display all contributions on the same graph. However, when $\varepsilon_W = \infty$, only the quantum tail $f^*_{\omega}(x, x; \varepsilon_W)$ exists. The corresponding coefficient $A^*(\tilde{x}, \tilde{x})$ is shown in Fig. 3.

5. TOWARDS THE DESCRIPTION OF THE QUANTUM SYSTEM

For the sake of simplicity, we still consider a One-Component Plasma in the following. The model of two quantum charges embedded in a classical OCP in the vicinity of an ideal conductor is expected to mimic the behavior of correlations in a fully quantum but rather weakly degenerate

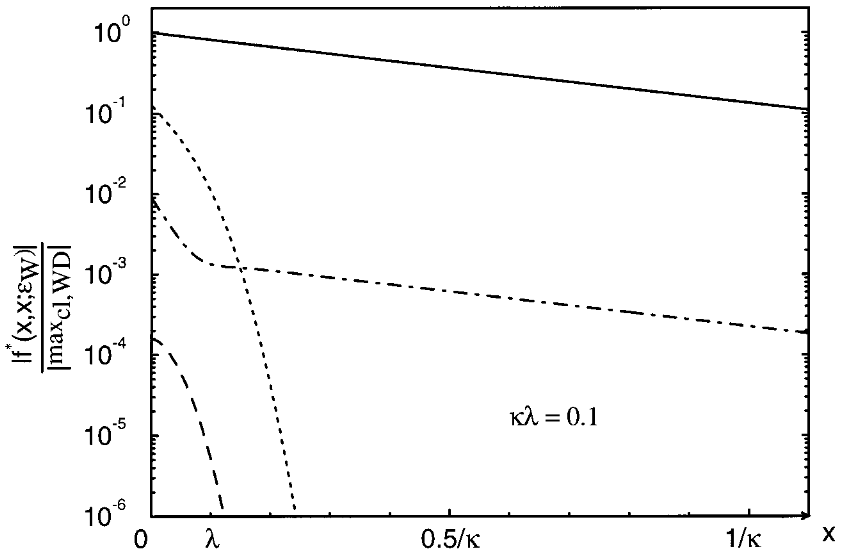

Fig. 2. Absolute values of various terms in the $\hbar$-expansion of f^*. The solid line is the classical term, $f_{cl,DW}$. The dotted line is the $\hbar$ term given by Eq. (4.19a), which is a purely diffraction term. The dash-dot line is the contribution (4.19b) at order $\hbar^2$ from diffraction effects, while the dashed line corresponds to the purely quantum term (4.21) at the same order.

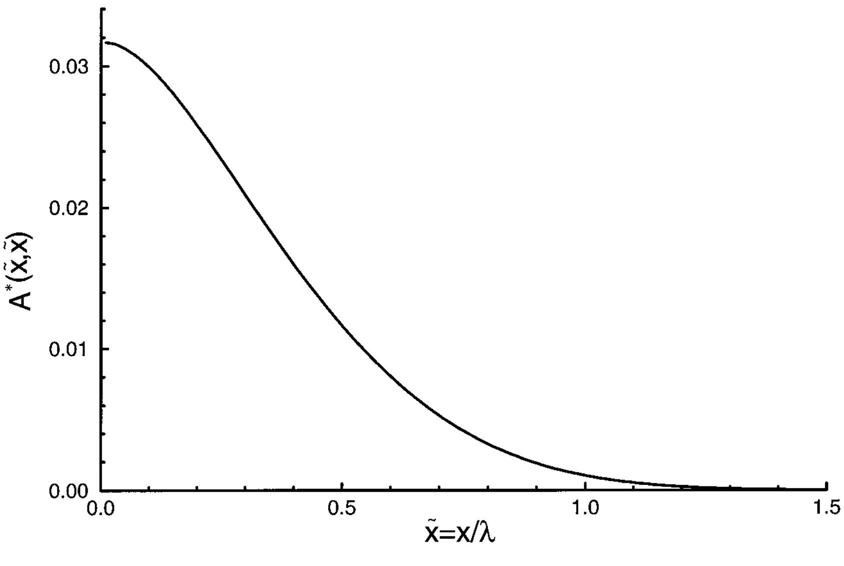

Fig. 3. Universal coefficient $A^*(\tilde{x}, \tilde{x})$ in the purely quantum $1/y^3$ tail (4.21). Only this part of the tail survives in the case of an ideally conductive wall.

OCP, where the de Broglie wavelength λ is equal to the one of the external charges in the model and the Debye length is $\xi = \xi_B$, with ξ_B defined after Eq. (3.1). The two conditions $a \ll \xi$ and $\lambda \ll \xi$ used to obtain explicit expressions when $\varepsilon_W = \infty$ are compatible with the low-degeneracy condition $\lambda \ll a$. The inequality $a \ll \xi$ assumed in Section 3 only implies that the Coulomb coupling is weak, in which case the condition $\lambda \ll \xi$ used in Section 4 is sufficient to obtain an explicit expression for the purely quantum part of the $1/y^3$ tail. [We notice that the condition $\lambda \ll \xi$ does not necessarily enforce the semiclassical condition in the weak-coupling regime, $\lambda \ll \Gamma a$ (see Section VI B of ref. 16 for more details).] On the contrary, in the case of a plain or dielectric wall, the explicit expressions of the $1/y^3$ tail in Section 4.2 are obtained only in the regime $a \ll \lambda_i \ll \xi_B$, which is not a weakly-degenerate situation. Subsequently, in the following, we will restrict our study to a OCP in the vicinity of an ideal conductor in the weak-coupling and low-degeneracy limit (1.13).

When exchange effect are negligible, and according to the formalism of ref. 17, the quantum truncated two-body distribution function

$$\rho^{(2)\,T}(\mathbf{r}, \mathbf{r}') \equiv \left\langle \sum_i \delta(\mathbf{r}_i - \mathbf{r}) \sum_{j \neq i} \delta(\mathbf{r}_j - \mathbf{r}') \right\rangle - \rho(\mathbf{r})\, \rho(\mathbf{r}') \tag{5.1}$$

where $\langle \cdots \rangle$ denotes the statistical ensemble average, may be obtained from the density $\rho_W(x, \boldsymbol{\xi})$ and the Ursell function $h_W(x, x', \mathbf{y}; \boldsymbol{\xi}, \boldsymbol{\xi}')$ in a classical gas of closed filaments with random Brownian shapes $\lambda\boldsymbol{\xi}$ and which interact via the two-body potential given by the Feynman–Kac formula. The relation reads[17]

$$\rho^{(2)\,T}(\mathbf{r}, \mathbf{r}') = \int D^0_W(\boldsymbol{\xi}; x) \int D^0_W(\boldsymbol{\xi}'; x')\, \rho_W(x, \boldsymbol{\xi})\, \rho_W(x', \boldsymbol{\xi}') \times h_W(x, x', \mathbf{y}; \boldsymbol{\xi}, \boldsymbol{\xi}') \tag{5.2}$$

According to its definition $\rho_W(x, \boldsymbol{\xi}) \equiv \langle \sum_i \delta(\mathbf{r}_i - \mathbf{r})\, \delta(\boldsymbol{\xi}_i - \boldsymbol{\xi}) \rangle$, the density $\rho_W(x, \boldsymbol{\xi})$ near the wall is related to its bulk value $\rho_{\text{bulk}}(\boldsymbol{\xi})$ through

$$\rho_W(x, \boldsymbol{\xi}) = \rho_{\text{bulk}}(\boldsymbol{\xi})\, e^{-\beta(F_{\text{elect},\, W}[q] - F_{\text{elect, bulk}}[q])} \tag{5.3}$$

where $F_{\text{elect},\, W}[q] (F_{\text{elect, bulk}}[q])$ is the classical immersion free energy of the filament defined by a formula analogous to Eq. (2.18) where $\langle \cdots \rangle_{U_0}$ is replaced by a Maxwell–Boltzmann average for the gas of filaments.

In the low-density regime (1.13), where a weak-coupling restriction is added to the low-degeneracy condition, according to the analysis of ref. 16,

$$\rho_{\text{bulk}}(\xi) = \rho \times [1 + \text{`}O(\rho)\text{'}] \tag{5.4}$$

where ρ is the quantum density of bulk particles and '$O(\rho)$' is a term of order ρ which comes from various effects. More precisely, these are coupling corrections of order Γ^3, diffraction effects of order $\Gamma(\lambda/a)^2$ and exchange contributions of order $(\lambda/a)^3$. All these terms are indeed negligible with respect to 1.On the other hand, under the weak-coupling hypothesis, the electrostatic immersion free energies are given by their Debye approximations which are of order $\Gamma^{3/2}/\beta$, as in Eq. (3.18). An extra expansion can be performed because the condition $\lambda \ll \xi = 1/\kappa$ is fulfilled according to Eq. (1.13). Then

$$e^{-\beta(F_{\text{elect, }W}[q] - F_{\text{elect, bulk}}[q])} = [1 + O(\Gamma^{3/2} f(\kappa x))][1 + O(\Gamma^{3/2} \kappa\lambda F(\kappa x, \xi))] \tag{5.5}$$

where $O(\Gamma^{3/2} f(\kappa x))$ comes from coupling corrections in the classical expression $F_{\text{cl, }W}(e) - F_{\text{cl, bulk}}(e)$ while $O(\Gamma^{3/2}(\kappa\lambda)\, F(\kappa x, \xi))$, which is of order $\Gamma^2(\lambda/a)$, originates from quantum corrections to the previous classical coupling terms. Combination of Eqs. (5.3), (5.4), and (5.5) shows that, at leading order in Γ,

$$\rho_W(x, \xi) \sim_{\Gamma \ll 1} \rho \tag{5.6}$$

In the regime (1.13), a phenomenological correlation $\rho^{(2)\,T}_{\text{phen}}(x, x', \mathbf{y})$ may be built as follows: one replaces the large-distance behavior of h_W in Eq. (5.2) by $-\beta$ times the limit of the effective interaction $\Phi^{\text{ext}}(x, x', y; \xi, \xi')$ between two external classical filaments embedded in a classical plasma. The effective interaction is defined by a generalization of Eq. (2.7) and the limit is calculated in the regime $\lambda \ll a \ll \xi_B$ and by replacing ξ_B by ξ. We recall that, since we have been able to calculate the expression in this limiting case only for an ideally conducting wall, we will restrict our phenomenological model to $\varepsilon_W = \infty$. Then the $1/y^3$ tail originates only from the purely quantum part ω in the immersion free energy of two filaments written in the last line of Eq. (2.20). According to Eqs. (5.2) and (5.6),

$$\rho^{(2)\,T}_{\text{phen}}(x, x', \mathbf{y}) \sim_{|\mathbf{y}| \to \infty} -\beta\rho^2 \int D^0_W(\xi; x) \int D^0_W(\xi'; x')\, \omega(x, x', \mathbf{y}; \xi, \xi') \tag{5.7}$$

The definition (4.7) of $\bar{\mu}^{(1)}$ and the relation (4.6) between D^0_W and the corresponding normalized measure $\bar{D}^0_W$ allow to write the $1/y^3$ tail of Eq. (5.7) as

$$\rho^{(2)\,T}_{\text{phen}}(x, x', \mathbf{y}) \sim_{|\mathbf{y}| \to \infty} \rho^2 [1 - e^{-2x^2/\lambda^2}][1 - e^{-2x'^2/\lambda^2}] \frac{f^*_\omega(x, x'; \varepsilon_W = \infty)}{y^3} \tag{5.8}$$

where $f^*_\omega(x, x'; \varepsilon_W = \infty)$ is given in Eq. (4.12).

Anyway, even in the low-density limit, the exact amplitude $f_{\text{qu}}(x, x', \varepsilon_W)$ for the quantum many-body problem likely involves not only the direct interaction between the quantum particles at x and x', but also quantum interactions involving one or two other charges in the plasma. Since the model of two external quantum charges in a bath where all particles are classical cannot incorporate the latter coupling effects involving more than two quantum charges, an exact many-body calculation is required. This remark is supported by a comparison already made for bulk correlations derived either from the solvable model or from an exact calculation for the many-body problem in the low-density limit (1.13).[7] For instance, at the first order in density, the coefficient of the exact $1/r^6$ tail in the bulk contains corrections to the value predicted by the solvable model: the effective interaction is not only a direct effect but it is partially conveyed by one or two intermediate charges which are classically screened from one of the particles at x or x'. (However, the root of the $1/r^6$ tail is the same in both calculations.) In the presence of a uniform magnetic field, the leading $1/r^5$ tail derived from the model happens to coincide with the exact result, but this coincidence originates from compensations involving indirect interactions and seems to arise from the purely quantum-statistical origin of magnetic effects.

6. CONCLUSION

The model exhibits the following relation between the large-distance behaviors of correlations and the structure of polarization clouds. At classical thermal equilibrium, the static *bulk* distribution functions (with some proper truncations) decrease faster than any inverse power of the relative distances when the latter become large.[1] The so-called *exponential clustering* means that the average configurations of the particles in the plasma are such that all the multipoles of a set of particles plus its surrounding polarization cloud vanish.[5, 18] However, *in the vicinity of a wall*, this "perfect" screening is partially canceled by an *electrostatic-geometric* effect: the deformation of the polarization cloud. This deformation with respect to the "perfect" arrangement generates a nonvanishing mean dipole between a particle and its polarization cloud in the direction perpendicular to the wall, though the corresponding total mean charge still vanishes. When the wall is a plain wall the effect is purely geometric, whereas some surface polarization charge appears inside walls with electrical properties. The resulting effective dipole-dipole interaction is reflected in the algebraic $1/y^3$ fall-off of the position-position correlation in the direction parallel to the wall when the latter is a plain wall or a dielectric one.[13] However, in the vicinity of an ideally conductive boundary, the cloud generated by the influence phenomenon inside the conductive wall fits with the polarization cloud created by the plasma charges so perfectly that the effective interaction between the plasma particles decreases faster than any inverse power-law (as in the bulk).

Contrarily, in quantum regimes, screening is always *algebraic*, even in the bulk, because intrinsic quantum position fluctuations destroy the perfect arrangement of classical average configurations. In the bulk, rotational invariance and the harmonicity of the Coulomb potential specific to the definition (1.1) enforce a $1/r^6$ tail (in some sense, only the squared fluctuations of dipole–dipole interactions survive).[4, 19] On the other hand, in the vicinity of a wall the previous geometrical effect arises again and (non-squared) $1/y^3$ dipolar interactions survive after averaging over microscopic configurations with quantum fluctuations, even in the vicinity of an ideally conducting wall.

APPENDIX A

In this appendix we derive the expressions (3.9)–(3.10) and (3.11) for the immersion free energies in the classical weak-coupling limit. These values are derived from the expansion of the definitions (2.18) and (2.21) with respect to βq_i up to second order. The expansion is combined with a self-consistent mean-field approximation for the correlations in the classical plasma in the absence of the external charge distributions $q_i(\mathbf{r})$.

For any potential $v(\mathbf{r}, \mathbf{r}')$ the linear and quadratic terms in the expansion of $F^{(1)}_{\text{elect}}[q_i]$ with respect to $q_i(\mathbf{r})$ read

$$F^{(1)}_{\text{elect}}[q_i] = \int d\mathbf{r} \int d\mathbf{r}'\, q_i(\mathbf{r})\, v(\mathbf{r}, \mathbf{r}') \langle Q(\mathbf{r}')\rangle_{U_O} + \frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}'\, q_i(\mathbf{r})\, v(\mathbf{r}, \mathbf{r}')\, Q^{\text{ind}}(\mathbf{r}'; q_i) \qquad \text{(A1)}$$

In Eq. (A1) the expression of the charge density $Q^{\text{ind}}(\mathbf{r}'; q_i)$ induced by $q_i(\mathbf{r})$ is that given by the linear response theory,

$$Q^{\text{ind}}(\mathbf{r}'; q_i) = -\beta[\langle Q(\mathbf{r}')\, V[q_i]\rangle_{U_0} - \langle Q(\mathbf{r}')\rangle_{U_0} \langle V[q_i]\rangle_{U_0}]$$
$$= -\beta \int d\mathbf{r}^* \int d\mathbf{r}''[\langle Q(\mathbf{r}')\, Q(\mathbf{r}^*)\rangle_{U_0} - \langle Q(\mathbf{r}')\rangle_{U_0} \langle Q(\mathbf{r}^*)\rangle_{U_0}]$$
$$\times\, v(\mathbf{r}^*, \mathbf{r}'')\, q_i(\mathbf{r}'') \qquad \text{(A2)}$$

A similar scheme leads to

$$\Phi_{\text{elect}}[q_1, q_2] + \int d\mathbf{r} \int d\mathbf{r}'\, q_1(\mathbf{r})\, v(\mathbf{r}, \mathbf{r}')\, q_2(\mathbf{r}')$$
$$= \tfrac{1}{2} \int d\mathbf{r} \int d\mathbf{r}'[q_1(\mathbf{r})\, Q^{\text{ind}}(\mathbf{r}'; q_2) + q_2(\mathbf{r}')\, Q^{\text{ind}}(\mathbf{r}; q_1)]\, v(\mathbf{r}, \mathbf{r}') \qquad \text{(A3)}$$

In the self-consistent meanfield approximation, the basic assumption has been formulated in Sec. 3.1 in terms of the total potential ϕ_W created by a charge and its polarization cloud. (It is equivalent to approximate the exact direct correlation function by $-\beta e^2 v(\mathbf{r}, \mathbf{r}')$, as mentioned in ref. 13.) ϕ_{MFW} is the solution of the integral Eq. (3.6). Comparison of Eq. (A2) with Eqs. (3.3), (3.5), and (3.6) shows that

$$Q^{\text{ind}}_{MF}(\mathbf{r}') = -\beta e^2 \rho(x') \int d\mathbf{r}''\, \phi_{MFW}(\mathbf{r}', \mathbf{r}'')\, q_i(\mathbf{r}'') \qquad \text{(A4)}$$

In the case of the Coulomb potential, the second term in Eq. (A1) can be simplified by insertion of the meanfield expression (A4) and use of a symmetry property. Indeed, by definition $\phi_{MFW}(\mathbf{r}, \mathbf{r}')$ is real while, according to the equations (3.7) and

$$\varepsilon_W \Delta_{r'} \phi_{MFW}(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}') \qquad \text{for} \quad x' < 0 \qquad \text{(A5)}$$

together with the boundary conditions, $\phi_{MFW}(\mathbf{r}, \mathbf{r}')$ is also the Green function of a self-adjoint operator acting on $\mathbf{r}'$. Thus $\phi_{MFW}(\mathbf{r}, \mathbf{r}')$ is a symmetric function of its arguments, as well as $v_C(\mathbf{r},\mathbf{r}')$,

$$\phi_{MFW}(\mathbf{r}, \mathbf{r}') = \phi_{MFW}(\mathbf{r}', \mathbf{r}) \qquad \text{(A6)}$$

Therefore

$$\int d\mathbf{r}'\, \rho(x')\, v_{CW}(\mathbf{r}, \mathbf{r}')\, \phi_{MFW}(\mathbf{r}', \mathbf{r}'')$$
$$= \int d\mathbf{r}'\, \rho(x')\, \phi_{MFW}(\mathbf{r}'', \mathbf{r}')\, v_{CW}(\mathbf{r}', \mathbf{r}) \tag{A7}$$

and by using again Eq. (3.6) together with Eq. (A4), we obtain

$$\int d\mathbf{r} \int d\mathbf{r}'\, q_i(\mathbf{r})\, v(\mathbf{r}, \mathbf{r}')\, Q^{\text{ind}}_{MF}(\mathbf{r}'; q_i)$$
$$= \int d\mathbf{r} \int d\mathbf{r}''\, q_i(\mathbf{r})[\phi_{MFW}(\mathbf{r}, \mathbf{r}'') - v_{CW}(\mathbf{r}, \mathbf{r}'')]\, q_i(\mathbf{r}'') \tag{A8}$$

Similarly, the combination of the latter equation with Eq. (A3) leads to the value (3.11) for the effective electrostatic interaction between two charge distributions.

APPENDIX B

In this appendix, we calculate the first and second moments of a Brownian bridge in the vicinity of a wall. In fact the value of $\bar{\mu}^{(1)}(\tilde{x})$ can be directly inferred from Eq. (2.15) of ref. 20. Two properties are used.

First, the link between the path integral and the Heisenberg representation in imaginary time reads

$$\frac{1}{(2\pi\lambda^2)^{1/2}} \int D^0_W([\xi]_x; x_1)\, \lambda[\xi(s)]_x \equiv \langle x_1 |\, e^{-\beta[\hat{h}_0 + U_W(\hat{x})]}[\hat{x}(s) - x_1]\, | x_1 \rangle \tag{B1}$$

where $\hat{h}_0$ is defined in Eq. (2.13). In Eq. (B1) $\hat{x}(s)$ is an operator in Heisenberg representation

$$\hat{x}(s) \equiv e^{\beta s[\hat{h}_0 + U_W(\hat{x})]}\hat{x}e^{-\beta s[\hat{h}_0 + U_W(\hat{x})]} \tag{B2}$$

By inserting the closure relation, we get

$$\langle x_1 |\, e^{-\beta[\hat{h}_0 + U_W(\hat{x})]}[\hat{x}(s) - x_1]\, | x_1 \rangle$$
$$= -x_1 g_{0,W}(x_1, x_1, 1) + \int_0^{+\infty} dx'\, g_{0,W}(x_1, x', 1-s)\, x' g_{0,W}(x', x_1, s) \tag{B3}$$

Second, $g_{0,W}(x', x_1, s)$ is merely given by the image method.[20] Indeed, we look for a function that obeys the same equation as the bulk solution with the only difference that it vanishes on the wall and it is set equal to zero inside the wall as soon as x or x' becomes negative. The function which obeys the last property and which reads

$$g_{0,W}(x, x'; s) = g_{0,\text{bulk}}(x - x'; s) - g_{0,\text{bulk}}(x + x'; s) \tag{B4}$$

for $x > 0$ and $x' > 0$ clearly satisfies all requirements. In Eq. (B4), the bulk solution reads

$$g_{0,\text{bulk}}(x - x'; s) = \frac{1}{(2\pi\lambda^2)^{1/2}} \frac{\exp[-|x - x'|^2/(2\lambda^2 s)]}{\sqrt{s}} \tag{B5}$$

By combination of Eqs. (B1) and (B3) with Eq. (B4) and by using the definition of the complementary error function recalled in Eq. (4.11), we get

$$\begin{aligned}\int D_W^0([\xi]_x; x_1)[\xi(s)]_x = \tilde{x}_1 \Bigg[2se^{-2\tilde{x}_1^2} - \operatorname{Erfc}\left(\frac{1}{\sqrt{2s(1-s)}}\,\tilde{x}_1\right) \\ + (1-2s)\, e^{-2\tilde{x}_1^2} \operatorname{Erfc}\left(\frac{1-2s}{\sqrt{2s(1-s)}}\,\tilde{x}_1\right)\Bigg]\end{aligned} \tag{B6}$$

where $\tilde{x} \equiv x/\lambda$. The integrals over s involved in the definitions of $\bar{\mu}^{(1)}$ and $\bar{\mu}^{(2)}$ are performed by means of an integration by parts (in order to eliminate the functions Erfc). Then the changes of variables $u = 1/\sqrt{2s(1-s)}$ for $0 \leqslant s \leqslant 1/2$ and next $v^2 = u^2 - 2$ are used. The final result is obtained from the representation

$$\operatorname{Erfc}(xy) = \frac{2x}{\pi}\, e^{-x^2y^2} \int_0^\infty dt\, \frac{e^{-t^2y^2}}{t^2 + x^2} \tag{B7}$$

which is valid when the real part of y^2 is strictly positive. Eventually, the value (4.9) of $\bar{\mu}^{(1)}(\tilde{x})$ is derived from (B6) and the normalization given in Eq. (4.6).

The same scheme is used for calculating $\int D_W^0([\xi]_x; x_1)[\xi(s)]_x^2$. The latter moment and the first one appear in the path integral representation of the matrix element

$$\frac{\sqrt{2\pi}}{\lambda} \langle x_1 |\, e^{-\beta[\hat{h}_0 + U_W(\hat{x})]} [\hat{x}(s)]^2 \,| x_1 \rangle \tag{B8}$$

According to the value (B4) of the free propagator, the matrix element is readily computed with the result

$$\tilde{x}_1^2[1-e^{-2\tilde{x}_1^2}(2s-1)^2]+s(1-s)(1-e^{-2\tilde{x}_1^2}) \tag{B9}$$

The value (4.10) of $\bar{\mu}^{(2)}(\tilde{x})$ is derived from (B9) in a straightforward way.

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